N62 13234

NASA TR R-113

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL REPORT
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1961

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. Yearly subscription, $15; foreign, $19; single copy price varies according to size - - - - - - - - - Price 40 cents

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Lewis Research Center
Cleveland, Ohio

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SUMMARY

Vehicle guidance during the midcourse phase of an interplanetary mission is analyzed. The basic guidance equations, in the form of deviations from a reference trajectory, are developed by linear perturbation methods. A self-contained measurement system is hypothesized, consisting of tracking devices to measure angles between pairs of celestial bodies, a clock, and instrumentation for thrust vector control. Measurement errors are specified by statistical distributions. The data-adjustment technique, which includes knowledge of the past history of the vehicle trajectory with current guidance information through a maximum-likelihood reduction, improves guidance knowledge successively as the vehicle progresses toward its target and thus has a significant effect on the efficiency of guidance maneuvers. The major results of data reduction (predicted miss distance at target and its estimated variance) are interpreted by logic or decision expressions that prescribe the amount of incremental velocity correction. Guidance logic is based on arbitrary dead-band and damping coefficients, varied parametrically for efficient guidance performance. Results indicate that small dead band and damping are desirable.

Guidance performance is evaluated by simulating the guidance maneuver on a digital computer and using Monte Carlo techniques of statistical analysis. The mission considered is a 192-day Earth-Mars transfer. An injection-velocity error of 70 feet per second rms is assumed, which would cause the vehicle to miss its target by several hundred thousand miles. Ten navigation fixes are taken en route, and the corrective action is specified by the guidance logic. With tracking-device errors of 10 seconds arc rms the largest total velocity requirement is about 235 feet per second, and the largest miss distance is 2000 miles. With increased sighting errors of 40 seconds arc rms, the velocity requirement increases to 355 feet per second and the miss distance to 4500 miles. The maximum numbers of corrections for these two cases are six and nine, respectively.

INTRODUCTION

It is recognized that simple ballistic trajectories will meet with little success in interplanetary flights. A study of errors by Ehricke (ref. 1) indicates that the accuracy required of the cutoff velocity at launch to result in position errors at the target planet of less than several thousand miles appears technically unfeasible. Therefore, it is reasonable to expect that a space vehicle will be equipped with a guidance system that will allow trajectory corrections en route.

The study of guidance maneuvers is conveniently divided into midcourse guidance and approach (terminal) guidance. The function of midcourse guidance is to ensure a successful rendezvous with the target planet, while that of approach guidance is to execute highly accurate maneuvers in the immediate vicinity of the planet (e.g., atmospheric-drag decelerations). Literature on approach guidance problems includes references 2 to 6.

This report is concerned with the guidance of a space vehicle during the midcourse phase of the mission. Some previous exploratory analyses of midcourse guidance have been reported (refs. 7 and 8). Reference 9 presents one of the few really comprehensive analyses and evaluations of a midcourse guidance theory. The present study investigates the use of data-adjustment and logic techniques to improve guidance efficiency and is an extension of the basic work of reference 9. These techniques have been developed and evaluated by the authors for the approach guidance problem (ref. 6) and have proved very satisfactory. The analysis considers the random sampling of real measurements with errors, and a multiple-correction guidance scheme. The statistical performance of the guidance maneuver is evaluated primarily on the basis of probability of final miss distance, total velocity-increment requirements, and number of corrections.

The cutoff velocity at launch (and consequently the heliocentric injection velocity) determines the interplanetary trajectory that will be followed. Errors in this quantity cause a deviation from the reference trajectory that must be corrected at a future time. If sensing equipment were perfect, the true deviation could be detected shortly after launch, the required velocity increment would be of the same order as the injection-velocity error, and no subsequent corrections would be necessary. However, in any real system, both the measurement of the deviation and the control of the corrective velocity increment will include random error inputs. Therefore, a single correction will not suffice unless it can be assured that the worst possible combination of errors incurred would still not cause the vehicle to miss its target by an amount greater than the specified mission requirement.

If the guidance system acts directly on measured deviations, and successive corrections are made, the vehicle will generally oscillate about the target trajectory with a corresponding wasting of fuel. That is, overcorrection may occur frequently because of the random nature of the measurement errors. A logical question, then, is what part of the information indicating deviations should be considered significant and acted upon. It is clearly indicated that a careful study must be made in order to deduce an optimum guidance scheme that would meet the guidance accuracy requirements, minimize the propulsion needed, and still not be impractical in terms of hardware specifications. This is essentially the problem of guidance in the face of random measurement errors.

The objective of this study is the formulation of a guidance theory that may be employed to prescribe efficient trajectory control. Consideration is given to a self-contained measurement system consisting of passive tracking devices to observe angles between selected pairs of celestial bodies, a clock to indicate time increments, and appropriate instrumentation for thrust vector control. It is assumed that Gaussian distributions are physically representative of system errors, and that calibration data are available.

Perturbation techniques are used to translate basic measurements to position deviations from a precomputed reference trajectory. At each point of data acquisition, a best estimate of the miss distance that will be incurred at the target is determined from a maximum-likelihood adjustment of current position deviations and predicted deviations obtained from past history of the vehicle trajectory. Knowledge is presumably increased in this way. An integral part of the guidance theory consists of logic or decision expressions used to prescribe the amount of velocity correction to be applied. The estimated variance of the miss distance serves as an indicator of the statistical uncertainty of available knowledge and thus is used to advantage in eleminating unnecessary or excessive corrections.

SYMBOLS

Note: Units of miles refer to statute miles (1 statute mile = 1.6093440 km).

Matrices:

A	(3×9) or (3×6) matrix defined by eqs. (B5), (C8)
B	(3×3) matrix defined by eq. (A31)
G	(3×3) matrix defined by eq. (B13)
H	(3×3) matrix defined by eq. (A27)
I	(3×3) unit matrix
M_A	(3×3) matrix defined by eq. (16)
M_o	(3×3) matrix of partial derivatives of $\overline{g}$ with respect to $\overline{r}_o$ (eq. (A5))
N	(3×3) matrix defined by eq. (B17)
P	(3×3) matrix defined by eq. (A28)
Q^*	(3×3) matrix of partial derivatives of components of $\overline{v}^*$ with respect to components of $\overline{r}$
R	(3×3) matrix of partial derivatives of $\overline{r}$ with respect to $\overline{r}_L$ (eqs. (A12), (A13))
R^*	(3×3) matrix of partial derivatives of $\overline{r}$ with respect to $\overline{r}_A$ (eqs. (A18), (A19))
U	(3×3) matrix whose rows are made up of vector $\overline{u}$ (eq. (8))

V (3×3) matrix of partial derivatives of $\bar{v}$ with respect to $\bar{r}_L$ (eqs. (A12), (A13))

V^* (3×3) matrix of partial derivatives of $\bar{v}$ with respect to $\bar{r}_A$ (eqs. (A18), (A19))

Σ (9×9) or (6×6) covariance matrix of derived measurements (eq. (B10))

Σ_1 (3×3) covariance matrix associated with $\delta\bar{r}_A$ (eqs. (C9), (B15), (B18))

$\Sigma_{1,2}$ (3×3) covariance matrix expressing correlation between $\delta\bar{r}_A$ and $\delta\bar{r}_{n-1}$

Σ_2 (3×3) covariance matrix associated with $\delta\bar{r}_{n-1}$ (eq. (B14))

Σ_3 (3×3) covariance matrix associated with $\delta\bar{r}_n$ (eqs. (B11), (B12), (B13))

Vectors:

All vectors are to be considered column vectors unless otherwise specified. The magnitude of a vector is written without a bar.

$\delta\bar{b}$ miss distance, miles

$\bar{c}$, $\bar{c}^*$ constants during coast, included in general solution of perturbed differential equations (eqs. (A22), (A23))

$\bar{\mathscr{E}}$ discrepancy vector in data adjustment (eq. (B3))

$\bar{g}$ acceleration due to gravity field

$\bar{i}_B$ unit vector in direction of $\bar{r}_B$

$\bar{i}_r$ unit vector in direction of $\bar{r}$

$\bar{i}_s$ unit vector in direction of star

$\bar{q}$ defined by eq. (13)

$\bar{r}$ heliocentric position of space vehicle, miles

$\Delta\bar{r}$ error in $\delta\bar{r}$ (eq. (15)), miles

$\delta\bar{r}$ position deviation from reference trajectory, miles

$\delta\bar{r}_A$ position deviation at reference time of arrival, miles

$\delta\bar{r}_A'$ desired position deviation at reference time of arrival determined by guidance logic (eq. (21)), miles

$\bar{r}_B$ distance from vehicle to planet sighted, miles

$\bar{r}_P$ heliocentric position of planet, miles

$\bar{u}$ defined by eq. (7) or (8)

$\bar{v}$ heliocentric velocity of space vehicle, miles/sec

$\bar{v}^*$ velocity required at position $\bar{r}$ to intercept target planet at t_A

$\Delta\bar{v}$ velocity increment due to impulsive thrust, ft/sec

$\delta\bar{v}$ velocity deviation from reference trajectory, miles/sec

$\bar{v}_P$ heliocentric velocity of planet, miles/sec

$\bar{v}_R$ velocity of vehicle relative to target planet at time t_A (hyperbolic velocity), ft/sec

$\Delta\bar{\alpha}$ error in $\delta\bar{\alpha}$, radians

$\delta\bar{\alpha}$ vector of angle deviations, radians

$\bar{\gamma}$ residual vector in data adjustment

$\bar{\lambda}$ Lagrange multiplier

Scalars:

Δb residual miss distance due to guidance logic (eq. (20)), miles

k_{DB} dead-band coefficient

k_{DM} damping coefficient

n integer number of correction

S quadratic form (eq. (B7))

t time, sec (days)

Δt error in t

t_A time of arrival

t_L time of departure

Δv_{max} maximum allowable velocity increment, ft/sec

Δv_{min} minimum allowable velocity increment, ft/sec

Δv_t total velocity increment $\left(\sum_{n=1}^{n_{max}} \Delta v_n\right)$, ft/sec

α reference angle between Sun and star (planet), radians

σ standard deviation of statistical distribution

σ_G rms error of measured miss distance (eq. (18)), miles

Subscripts:

A evaluated at t_A

L evaluated at t_L

n evaluated at t_n

x,y,z components of vector

Superscripts:

T transpose of a matrix

-1 inverse of a matrix

$\circ$ refers to measured quantity

ANALYSIS

The problem under consideration is that of guiding a space vehicle to intercept a moving planet at a given time and position in space. A typical interplanetary transfer is illustrated in figure 1. At the time of departure t_L, a velocity change is imparted to the vehicle causing it to escape the gravitational field of the departure planet. The vehicle then coasts around the Sun along a trajectory designed to intercept the target planet at the reference time of arrival t_A. Func-

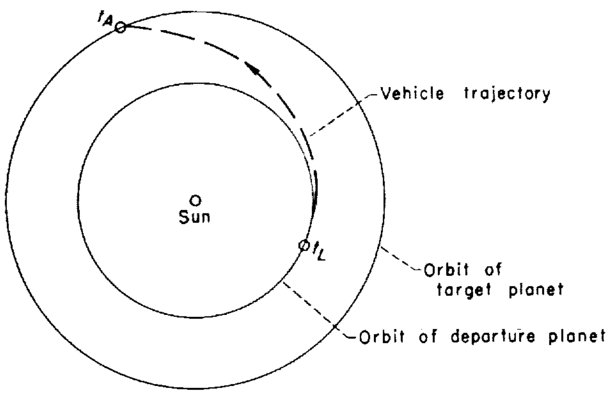


FIGURE 1.—Planar representation of interplanetary transfer.

tionally and for analysis purposes, it is convenient to divide the interplanetary transfer into three arcs: (1) in the vicinity of the departure planet, (2) in heliocentric space, and (3) in the vicinity of the target planet. This study will deal with the problem of vehicle guidance during the midcourse phase nominally defined as arc (2).

The following sections are concerned with (1) the basic linearized guidance equations that relate the correctional velocity increment to measured position deviations, (2) the measurement scheme used to determine the position deviations, (3) the technique of least-squares data adjustment, (4) the technique of guidance logic, and (5) the computational method used to evaluate guidance performance. In the following analysis it is assumed that the position deviation at time of launch is zero and that the perturbation on the trajectory is due only to deviation in the injection velocity. Furthermore, it is assumed that corrective velocity increments are impulsive in effect.

LINEARIZED GUIDANCE EQUATIONS

The current literature contains numerous references to the development of a linearized guidance theory. References 8 and 9 are particularly appropriate. The basic idea in this approach is the existence of a well-defined reference trajectory that the space vehicle should follow in the absence of all guidance errors. If it is assumed that at no time will the actual trajectory deviate significantly from the reference trajectory, then it is possible to study these deviations and the required correctional maneuvers by perturbation methods. The complete analysis is presented in appendixes A and C, and the major results are indicated here.

The geometry of a correctional maneuver is schematically shown in figure 2. At time t_n the vehicle, through a set of measurements as yet

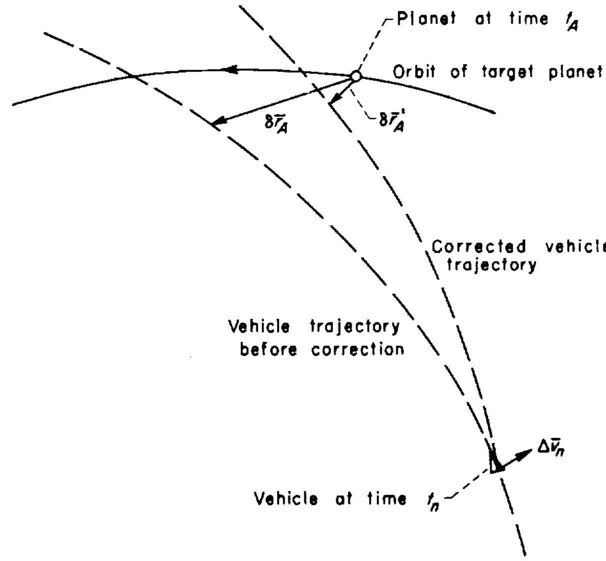


FIGURE 2.—Geometry of midcourse correction.

undefined, determines its position deviation at the reference time of arrival to be $\delta\bar{r}_A$. A velocity increment $\Delta\bar{v}_n$ is applied, causing the vehicle to follow a new trajectory with a resultant position deviation at t_A equal to $\delta\bar{r}'_A$. Appendix A shows that the predicted position deviation $\delta\bar{r}_A$ and the required velocity increment are linear functions of two position deviations $\delta\bar{r}_{n-1}$ and $\delta\bar{r}_n$ measured along the trajectory at times t_{n-1} and t_n, respectively ($t_{n-1} < t_n$). All deviations referred to are with respect to the reference trajectory. The specific relations are

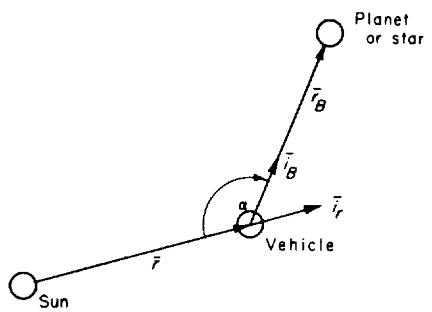
$$\delta\bar{r}_A = (B_n P_n)\delta\bar{r}_{n-1} - (B_n H_n)\delta\bar{r}_n \tag{1}$$

$$\Delta\bar{v}_n = B_n^{-1}\delta\bar{r}'_A + H_n\delta\bar{r}_n - P_n\delta\bar{r}_{n-1} = B_n^{-1}(\delta\bar{r}'_A - \delta\bar{r}_A) \tag{2}$$

where the quantities B_n, H_n, and P_n are matrices whose elements are evaluated along the reference trajectory at times t_{n-1} and t_n. These matrices, which may be considered precomputed quantities, are expressed in greater detail by equations in appendix A. It is to be noted that equations (1) and (2), while they are derived from linear perturbations on the differential equations of motion, are not restricted to the classical two-body problem. It is theoretically possible to include the simultaneous effect of all relevant gravitational fields, as indeed would be done in a final analysis. However, since the purpose of this study is to present and evaluate a guidance technique, the approximation of three-dimensional Keplerian orbits is felt valid and is used to generate the reference trajectory and the matrices of interest.

MEASUREMENT SCHEME

The preceding section has established the requirement of determining position deviations. One possible method of obtaining the components of $\delta\bar{r}(t)$ is now presented. Consideration is given to a self-contained measurement system consisting of optical tracking devices to observe angles between three selected pairs of celestial bodies, and a clock to indicate the reference time for the sighting process to begin. Specifically, the Sun, two stars, and a planet will be treated in this analysis, the Sun being used in all measurements. The geometry of a typical sighting is illustrated by sketch (a).



(a)

where $\bar{i}_r$ and $\bar{i}_B$ are unit vectors in $\bar{r}$ and $\bar{r}_B$ directions. From the dot product,

$$\bar{r}\cdot\bar{r}_B = -rr_B\cos\alpha \tag{3}$$

The sighting is taken at time t as indicated by the spaceship clock, and assume for the moment that no error is involved in time and that all sightings are instantaneous. Taking differentials of (3),

$$\delta(\bar{r}\cdot\bar{r}_B)=\bar{r}\cdot\delta\bar{r}_B+\bar{r}_B\cdot\delta\bar{r} \tag{4}$$

$$\begin{aligned}\delta(-rr_B\cos\alpha)&=(rr_B\sin\alpha)\delta\alpha-(r_B\cos\alpha)\delta r\\&\quad-(r\cos\alpha)\delta r_B\\&=(rr_B\sin\alpha)\delta\alpha-r_B(\bar{i}_r\cdot\delta\bar{r})\cos\alpha\\&\quad-r(\bar{i}_B\cdot\delta\bar{r}_B)\cos\alpha\end{aligned} \tag{5}$$

Equating (4) and (5), simplifying, and noting that $\delta\bar{r}_B=-\delta\bar{r}$,

$$\delta\alpha=\left[\frac{1}{r\sin\alpha}(\bar{i}_B+\bar{i}_r\cos\alpha)-\frac{1}{r_B\sin\alpha}(\bar{i}_r+\bar{i}_B\cos\alpha)\right]\cdot\delta\bar{r} \tag{6}$$

For a given reference trajectory and time of fix, the angle α and the quantity within brackets are predetermined. Since $\delta\alpha$ is simply the difference between measured and predetermined α, the components of $\delta\bar{r}$ may be obtained from three equations of the type (6). When one body is a star, r_B is infinite for all practical purposes, and the second term within the bracket drops out. To simplify notation, the bracketed term is denoted by the vector $\bar{u}$. Thus,

For Sun-star measurements:

$$\bar{u}=\frac{1}{r\sin\alpha}(\bar{i}_s+\bar{i}_r\cos\alpha) \tag{7}$$

For Sun-planet measurements:

$$\bar{u}=\left[\frac{1}{r\sin\alpha}(\bar{i}_B+\bar{i}_r\cos\alpha)-\frac{1}{r_B\sin\alpha}(\bar{i}_r+\bar{i}_B\cos\alpha)\right] \tag{8}$$

A set of equations (6) for measurements $\delta\alpha_i$ $(i=1,3)$ can be expressed in matrix notation

$$\delta\bar{\alpha}=U\delta\bar{r} \tag{9}$$

where $\delta\bar{\alpha}$ is a column vector and U is a (3×3) matrix whose rows comprise the row vectors $\bar{u}_i$. Premultiplying (9) by the inverse of U,

$$\delta\bar{r}=U^{-1}\delta\bar{\alpha} \tag{10}$$

providing U is not singular, which is generally true.

The effect of errors in the primary measurements $\delta\alpha_i$ and t upon the error in determining $\delta\bar{r}$ will now be investigated. Measurement errors are assumed random and physically represented by Gaussian (normal) distributions. These distributions will be considered in greater detail later. The random error in measuring α_i is denoted $\Delta\alpha_i$, and the random clock drift is denoted Δt. All random errors are defined here as the difference between measured and actual quantities. Thus, a positive Δt represents a fast clock. The error in $\delta\bar{r}$, denoted by $\Delta\bar{r}$, is due to the following three major effects:

(a) The effect of $\Delta\alpha_i$

(b) The motion of the planet sighted between reference and actual times of sighting, Δt

(c) The motion of the vehicle during this time interval

The component $\Delta\bar{r}_1$ due to (a) is found from equation (10):

$$\Delta\bar{r}_1 = U^{-1}\Delta\bar{\alpha} \tag{11}$$

The component $\Delta\bar{r}_2$ due to (b) is found as follows: Let α_3 denote the Sun-planet angle. From (3),

$$-rr_B \cos\alpha_3 = \bar{r}\cdot\bar{r}_B$$

Taking differentials due to the planet's motion and substituting $\delta\bar{r}_B = -\bar{v}_P\,\Delta t$, where $\bar{v}_P$ is the planet's velocity,

$$\delta\alpha_3 = \left[-\frac{1}{r_B \sin\alpha_3}(\bar{i}_r + \bar{i}_B\cos\alpha_3)\cdot\bar{v}_P\right]\Delta t \tag{12}$$

Equation (12) expresses the deviation in α_3 from the reference value due to the motion of the planet in the time interval Δt. From (10), the component $\Delta\bar{r}_2$ may be written in general form

$$\Delta\bar{r}_2 = U^{-1}\bar{q}\Delta t;\quad \bar{q}\equiv\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{13}$$

where $q_1 = q_2 = 0$ if α_1 and α_2 are Sun-star angles, and q_3 is equal to the bracketed term in equation (12).

The component $\Delta\bar{r}_3$ due to (c) is simply the product of vehicle velocity $\bar{v}$ and the clock error Δt:

$$\Delta\bar{r}_3 = -\bar{v}\Delta t \tag{14}$$

The total error is the sum of equations (11), (13), and (14):

$$\Delta\bar{r} = U^{-1}\Delta\bar{\alpha} + (U^{-1}\bar{q} - \bar{v})\Delta t \tag{15}$$

A few general observations concerning the sensitivity of $\Delta\bar{r}$ to $\Delta\bar{\alpha}$ can be made. Each set of linear equations of the type (9) relating position to angle deviations represents a plane. If minimum sensitivity to errors $\Delta\alpha_i$ is desired, the three planes should be mutually orthogonal. This criterion leads to the result that the vectors $\bar{u}_i$ be mutually orthogonal. For Sun-star measurements, $\bar{u}_s\cdot\bar{i}_r = 0$, so that $\bar{u}_{s_1}$ and $\bar{u}_{s_2}$ are both orthogonal to the Sun-vehicle line. Then, the condition that $\bar{u}_{s_1}$ and $\bar{u}_{s_2}$ be orthogonal requires that the planes in which the Sun-star angles are measured be orthogonal. For vehicles traveling near the ecliptic, a sufficient condition is that one star lie in the ecliptic while the other lie in a perpendicular plane. The final orthogonality condition is that $\bar{u}_P$ (eq. (8)) lie in the Sun-vehicle line. It can be shown that this criterion requires the Sun-planet line to be perpendicular to the vehicle-planet line. Such a condition cannot be realized in general; however, a judicious choice of the planet observed is indicated. The best choice of celestial bodies to use for a given navigation fix is a study in itself and therefore will not be pursued further here.

No attempt has been made in this section to include the effects of (1) finite speed of light, (2) time elapsing between beginning and end of sighting process, and (3) inaccuracy in the astronomical unit. The first two effects are essentially known, and as such they may be included as stored constants in the spaceship computer and the navigation fix corrected accordingly. The effect of (3), however, can influence guidance accuracy and should be accounted for in a final analysis. It is expected that the error due to the astronomical unit will be relatively small in comparison with the major navigation errors due to practical sensing devices.

DATA-ADJUSTMENT TECHNIQUE

The predicted position deviation at arrival $\delta\bar{r}_A$ can be computed from equation (1) given two measurements of position deviation along the path, $\delta\bar{r}_{n-1}$ and $\delta\bar{r}_n$, obtained at times t_{n-1} and t_n, respectively. Assume current data acquisition is at time t_n. Equation (2) then yields the desired velocity increment required to accomplish a given correctional manuever. Since the quantities $\delta\bar{r}_{n-1}$ and $\delta\bar{r}_n$ include the effect of random measurement errors, the prediction of $\delta\bar{r}_A$ will also be in error and will consequently affect any guidance action attempted. That is to say, there exists a significant probability that the amount of Δv called for will be in excess of the ideal requirement, and furthermore the position deviation at arrival after the correction may actually be increased, resulting in wasteful guidance action. Since random measurement errors cannot be wished away, what is desired then is a means of improving the accuracy of the prediction by some smoothing process. This will result in more efficient guidance action in terms of guidance accuracy and fuel expenditure.

The form of data adjustment used in a guidance theory must be arrived at by considering the requirements on the complexity and operating conditions of sensing-device systems and on computer capability. One form of adjustment proposed in reference 9 consists of measuring more than the minimum number of angles required for a navigation fix and then taking advantage of the redundancy of information by a simple least-squares fit, thereby obtaining a better estimate of position deviations. This leads to a satisfactory improvement for a given navigation fix, but gives no assurance that the accuracy of guidance knowledge will be improved with each successive fix. A method that assures that guidance action based on information at t_n will be more accurate than such action at t_{n-1} is a desirable result, and one to be pursued here.

The data-adjustment technique adopted for this study is similar to one that has proved very satisfactory in an investigation of approach guidance as reported by the authors (ref. 6). This technique consists of including knowledge of the past history of the vehicle trajectory along with current guidance information, and reducing the redundant information by an application of statistical theory. In clarification, the quantity $\delta\bar{r}_{n-1}$ is in a sense past history; however, what is specifically referred to here is the quantity $\delta\bar{r}_A$, which is a parameter that adequately describes the end conditions of the trajectory and is invariant during coasting periods. For example, after guidance action is taken at time t_{n-1}, the vehicle has an estimate of the resulting position deviation at arrival; from equation (2) this estimate is $\delta\bar{r}'_A$. This knowledge is then propagated to the next guidance point at t_n, since the vehicle has coasted during the interval. Consequently, if a statistical measure of the accuracy of this knowledge is available (which is the case), then $\delta\bar{r}_A$ may be treated as an indirectly measured quantity in equation (1), resulting in redundant information. An analogy can be made to the classical problem of measuring three angles of a triangle independently, although any two angles would suffice, since the sum must equal 180°.

The method of statistical adjustment of data has been reported extensively in the literature (e.g., refs. 10 to 12). In reference 11, Brown extends the work of earlier investigators and solves the general problem of "least-squares" adjustment considering correlated errors having the general multivariate normal distribution (see eq. (B6)). Since the primary measurement errors ($\Delta\alpha_i$, Δt) considered here are assumed to be independent and normally distributed, the indirectly measured quantities ($\delta\bar{r}_A$, $\delta\bar{r}_{n-1}$, $\delta\bar{r}_n$) meet the preceding criterion. Brown's results may therefore be applied to the data-adjustment problem in this study. The technique should properly be termed the "maximum-likelihood adjustment" because the term "least squares" corresponds only to the case of uncorrelated errors. The detailed development of the adjustment is presented in appendixes B and C, and the major results are indicated in this section.

Denoting $\delta\bar{r}^\circ_A$, $\delta\bar{r}^\circ_{n-1}$, and $\delta\bar{r}^\circ_n$ as the indirectly measured quantities, the adjusted quantities can be expressed as

$$\delta\bar{r}_A=\delta\bar{r}^\circ_A+\bar{\gamma}_1$$

$$\delta\bar{r}_{n-1}=\delta\bar{r}^\circ_{n-1}+\bar{\gamma}_2$$

$$\delta\bar{r}_n=\delta\bar{r}^\circ_n+\bar{\gamma}_3$$

where the $\bar{\gamma}$ are the most probable correction terms, called residuals, and are given by the following matrix equation:

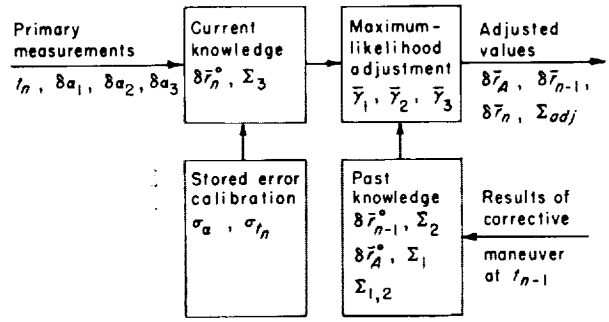
$$\bar{\gamma}=\begin{Bmatrix}\bar{\gamma}_1\\ \bar{\gamma}_2\\ \bar{\gamma}_3\end{Bmatrix}=-\Sigma A^T(A\Sigma A^T)^{-1}\bar{\mathscr{E}}$$

The matrix A (eq. (B5)) is a predetermined constant for a given reference trajectory and time of fix. The discrepancy vector $\bar{\mathscr{E}}$ (eq. (B3)) is evaluated at the measured conditions and thus depends in part upon random errors. The covariance matrix Σ is in effect a measure of the accuracy of past and present knowledge before adjustment, and also of the correlation between the indirectly measured quantities. The second major result is the covariance matrix of the adjusted quantities:

$$\Sigma_{adj}=\Sigma-[(A\Sigma)^T(A\Sigma A^T)^{-1}(A\Sigma)]$$

By comparing Σ_{adj} with Σ, one may gage the improvement due to the adjustment. Qualitatively, knowledge always improves, since the diagonal elements of Σ and the bracketed matrix are always positive as they represent the variances (mean-square error) in the principal coordinate axes (x,y,z).

Information flow related to the adjustment procedure may be shown schematically by the block diagram in sketch (b).



(b)

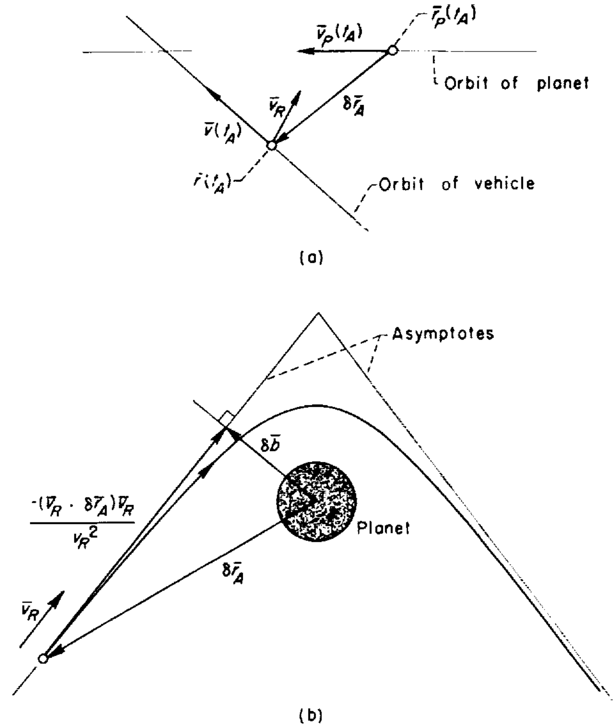
The covariance submatrices Σ_1, Σ_2, Σ_3, and $\Sigma_{1,2}$ make up the covariance matrix Σ. These (3×3) submatrices are associated with the quantities $\delta\bar{r}_A^\circ$; $\delta\bar{r}_{n-1}^\circ$, $\delta\bar{r}_n^\circ$, and the correlation between $\delta\bar{r}_A^\circ$ and $\delta\bar{r}_{n-1}^\circ$, respectively:

$$\Sigma=\begin{bmatrix}\Sigma_1 & \Sigma_{1,2} & 0\\ \Sigma_{1,2} & \Sigma_2 & 0\\ 0 & 0 & \Sigma_3\end{bmatrix}$$

These submatrices are examined in detail in appendix B. It suffices to say here that the diagonal elements of $[\Sigma_1]_n$ are in general smaller than those of $[\Sigma_1]_{n-1}$, so that one may conclude that the error in predicting $\delta\bar{r}_A$ decreases with each successive navigation fix.

GUIDANCE LOGIC

Definition of miss distance.—In the preceding development, the target parameter has been defined as the position deviation (in heliocentric coordinates) at the reference time of arrival t_A. At this point the target parameter is better defined in terms of the hyperbolic approach trajectory relative to the target planet. It is recalled that the reference trajectory considered in this analysis is a direct intercept, so that the approach trajectory referred to here is actually the perturbed approach trajectory. Motion in the vicinity of the target planet is illustrated in figure 3. Figure 3(a) shows the approximate linear motion of the planet and vehicle in the vicinity of rendezvous. At time t_A the vector difference between vehicle and planet positions is denoted $\delta\bar{r}_A$ as before, and the vector difference between vehicle and planet velocities is denoted $\bar{v}_R$. The quantity $\bar{v}_R$ is the velocity relative to the planet and is commonly referred to as the hyperbolic velocity. Figure 3(b) shows the approach trajectory and the definition of miss



(a) Linearized heliocentric motion at time of arrival.
(b) Hyperbolic approach trajectory.
FIGURE 3.—Motion in vicinity of target planet.

distance. At a great distance from the planet the trajectory approaches the asymptote as shown. The vector $\delta\bar{b}$ is the component of position deviation normal to the direction of relative motion, and is sometimes referred to as the "asymptotic displacement" or "vector point of aim." This quantity is useful as a target parameter for guidance purposes, since it is invariant with small time variations from the reference time of arrival. Consequently, the miss distance vector will be defined as

$$\left.\begin{aligned}\delta\bar{b}&=\delta\bar{r}_A-\frac{(\bar{v}_R\cdot\delta\bar{r}_A)\,\bar{v}_R}{v_R^2}\\ &=\left(I-\frac{\bar{v}_R\bar{v}_R^T}{v_R^2}\right)\delta\bar{r}_A\\ &=M_A\delta\bar{r}_A\qquad \underline{M_A\text{ is singular}}\end{aligned}\right\}\qquad(16)$$

Since M_A is a matrix predetermined by the reference trajectory, the miss distance is expressed by (16) as a linear function of the heliocentric position deviation at t_A.

The adjusted convariance matrix associated with $\delta\bar{b}$ is related to that of $\delta\bar{r}_A$ by

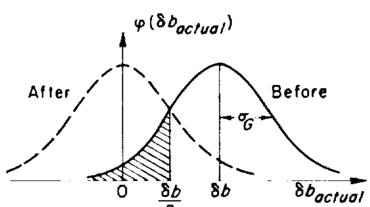
$$[\Sigma]_{\delta\bar{b}}=M_A\Sigma_1 M_A^T$$

$$=\begin{bmatrix}\sigma_x^2 & \sigma_{x,y}^2 & \sigma_{x,z}^2\\ \sigma_{y,x}^2 & \sigma_y^2 & \sigma_{y,z}^2\\ \sigma_{z,x}^2 & \sigma_{z,y}^2 & \sigma_z^2\end{bmatrix} \tag{17}$$

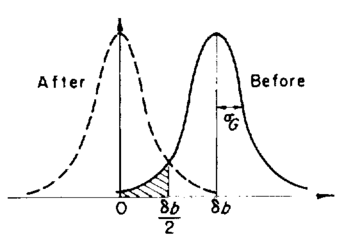
The off-diagonal elements are present because of the nonindependence of $\delta\bar{b}$ components. The accuracy criterion for midcourse guidance will be defined here as the magnitude of the miss-distance vector δb. It is assumed that directional errors will be corrected in the near vicinity of the planet where the navigation scheme can be planet-oriented and presumably more accurate. For the purpose of guidance logic, then, the rms error in the prediction of δb is denoted σ_G and defined

$$\sigma_G=\sqrt{\sigma_x^2+\sigma_y^2+\sigma_z^2} \tag{18}$$

Decision expressions.—As stated earlier, in the face of random measurement errors and successive corrections, the vehicle will tend to overcorrect (in a statistical sense) and therefore be wasteful of propellant. The seriousness of this inefficiency will of course depend on the size of the measurement errors. A qualitative understanding of the problem may be gained by considering the statistical density distribution of actual miss distance before and after a correction, shown



(c)



(d)

schematically by sketches (c) and (d). The distribution of δb_{actual} is shown centered about the predicted miss distance δb with a representative but not necessarily true shape. For discussion purposes, δb is always to the right of the origin, and those vehicles having δb_{actual} to the left of the origin can be considered as approaching the target from the opposite direction in a one-dimensional sense. The predicted miss distance is the same in sketches (c) and (d); however, sketch (d) represents a more accurate prediction because σ_G is smaller. Consistent with the definition of the density distribution, the total area under each curve is unity. The vehicle has no definite knowledge of δb_{actual} except that it probably lies within $\pm 3\sigma_G$ of δb with about a 99-percent probability.

If guidance action is taken to null δb, then the distribution of actual miss distance after the correction is centered about the origin, and the distribution's shape is essentially unchanged. It is clear that if $\delta b_{\ actual}<\frac{\delta b}{2}$, the vehicle will be moved farther from the target, thus causing guidance action to be wasted. The shaded area in sketches (c) and (d) is the probability of this occurrence. The conclusion is that the probability of wasted guidance action increases with increasing σ_G and with decreasing δb. Quantitatively, if the distribution were truly normal (Gaussian), the probability of overcorrection would be 40 percent for $\delta b=0.5\sigma_G$ and only 7 percent for $\delta b=3\sigma_G$. It seems logical that there is some limiting value of δb for which guidance action is not warranted, and that this value should be based on the rms error σ_G. A first guess would be to make this limiting value on the order of $3\sigma_G$, thereby making the probability of overcorrection small. Such a decision, however, may not lead to the minimization of total Δv, which is the criterion of efficient guidance action. The reasoning here is that those vehicles associated with large δb_{actual} must await correction at a later time when the Δv cost of such correction is conceivably much higher. One appreciates that the best decision is not obvious.

The use of a specific type of guidance logic to reduce inefficient use of Δv has been proposed and evaluated in a study of approach guidance, as reported by the authors in reference 6. The results indicated a significant reduction in the total

Δv and number of corrections required. This technique will now be included in the midcourse guidance theory, and its effect will be evaluated.

For a given navigation fix the vehicle computer is required to make the following decisions: (1) whether a correction is warranted, (2) the size of the correction, and (3) whether the Δv required is within the efficient limits of the propulsion devices. In the formulation of the decision expressions, common techniques used in control theory are adopted, namely dead band and damping. In addition, upper and lower limits on the magnitude of the velocity increment are specified. The lower limit will be seen to act essentially as an additional dead band. The specific guidance logic is shown in block-diagram form in figure 4. The logic is based on the results of data reduction, namely the adjusted values of $\delta \bar{r}_A$, δb, and σ_G.

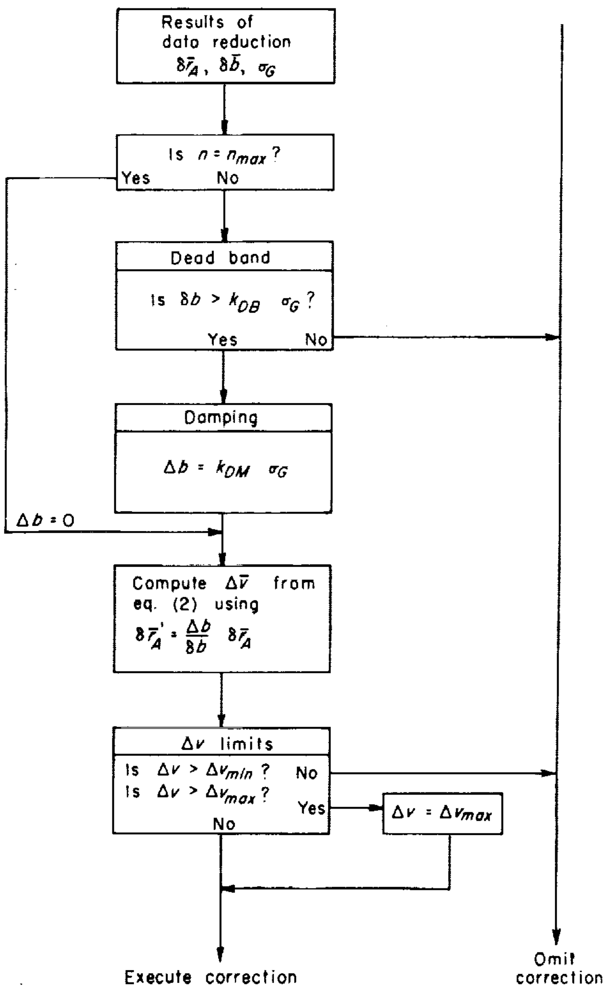


FIGURE 4.—Block diagram of guidance logic.

The normal logic is modified at the last correction point in order to ensure maximum guidance accuracy at the target.

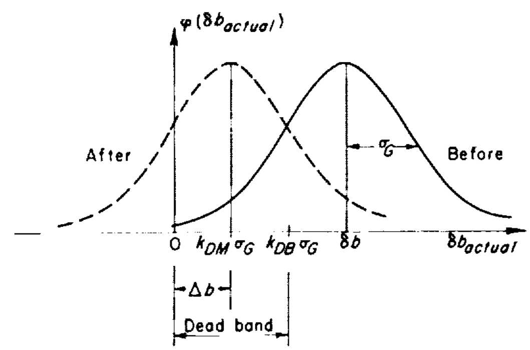
Consider first the use of dead band. The decision expression is an inequality that compares the predicted miss distance with a measure of statistical uncertainty in the prediction. An arbitrary dead-band coefficient k_{DB} is introduced so that desirable dead-band sizes may be determined by optimization of results. A corrective manuever is made only when

$$\delta b > k_{DB} \sigma_G \tag{19}$$

Once a correction is decided upon, damping action is used to reduce the probability of overcorrection. Instead of attempting to null δb, the vehicle is left with an intentional miss distance

$$\Delta b = k_{DM} \sigma_G \tag{20}$$

where the damping coefficient k_{DM} is introduced to permit optimization of results. Sketch (e) illustrates the damping action schematically in terms of the δb_{actual} density distribution. The



(e)

vehicle should probably be corrected within the dead band ($k_{DM} \leq k_{DB}$); and with damping, overcorrection occurs only if $\delta b_{actual} < \frac{\delta b}{2} - \frac{k_{DM} \sigma_G}{2}$. The probability of overcorrection would then be least when the dead-band and damping coefficients are equal.

The corrective velocity increment is computed from equation (2), using a desired position deviation vector at arrival $\delta \bar{r}_A'$, which has the same direction and is proportional to the predicted vector $\delta \bar{r}_A$. With reference to figure 2,

$$\delta \bar{r}_A' = \frac{\Delta b}{\delta b} \delta \bar{r}_A \tag{21}$$

From equation (2),

$$\Delta\bar{v}=B_n^{-1}\left(\frac{\Delta b}{\delta b}\,\delta\bar{r}_A-\delta\bar{r}_A\right)$$

$$=-\left(1-\frac{k_{DM}\sigma_G}{\delta b}\right)B_n^{-1}\delta\bar{r}_A \qquad (22)$$

Thus, the Δv magnitude for a damped correction is equal to the fraction $\left(1-\frac{k_{DM}\sigma_G}{\delta b}\right)$ of that required for the undamped correction. The use of thrust devices having limited Δv capability for individual corrective maneuvers is considered by supplying the guidance computer with the minimum and maximum Δv that are to be allowed. This limited capability may be due to excessive startup losses and control requirements on the low side, and engine cooling problems on the high side. If the Δv specified by the guidance logic is smaller than Δv_{min}, no correction is made. Alternatively, if an increment greater than Δv_{max} is called for, only Δv_{max} is supplied.

COMPUTATIONAL METHODS

Numerical results used in the evaluation of guidance performance are obtained with the aid of an IBM 704 digital computer. A representative interplanetary mission is chosen by specifying the target planet, launch date, and trip time. These quantities are considered input to a "reference trajectory program," which calculates a three-dimensional, Keplerian transfer ellipse. For this purpose, the planetary orbits are also considered Keplerian ellipses. With the reference trajectory completely defined, the perturbation matrices discussed in the previous sections and appendix A are evaluated at predetermined times t_n. The details of the calculation are not discussed in this report; however, standard methods such as presented in reference 9 are used.

Midcourse guidance is simulated through the use of Monte Carlo statistical techniques. This approach consists of generating a random injection velocity error, which then results in a random perturbed trajectory. At the reference time of fix, random errors are added to the true measurements, the data are adjusted by the method described, and the type of corrective action is determined by the guidance logic. Random errors are added to the incremental velocity vector, the true perturbed trajectory is calculated, and the process is repeated until the vehicle completes the guidance maneuver. Statistical results are developed from many random simulations of this type. Results include (1) number of corrections made, (2) size of individual velocity increment used, (3) total velocity increment used, (4) miss distance at the target, and so forth.

Random errors used herein are approximately normally distributed and are generated by a method described and evaluated in reference 13.

Use of the Monte Carlo technique for obtaining statistical results offers a number of advantages for vehicle guidance analysis. First, the number of random variables that could be considered is unlimited, and all may be taken into account simultaneously. A running statistical record of guidance performance is obtained, and the identity of a given vehicle need not be lost. The sampling technique precludes complex mathematical manipulation of previously known distribution types; this is principally applicable since the effect of guidance logic is intentionally to alter the distribution shape from any classical form it may have had.

The major disadvantage of Monte Carlo is that a very large number of samples may be needed to characterize the infinite sample accurately, and thus it may require excessive computing time. Average or probable events can be obtained with a fairly high degree of accuracy from a relatively small sample; however, unlikely occurrences may not be well defined. Although probable values are of interest, the worst requirements on the guidance system are of principal importance to design, since a high probability of success will probably be demanded for manned missions.

The significance of the maximum requirements from a given sample of an approximately continuous distribution can be estimated using statistics of extreme values (ref. 14). The mean number of exceedances X_m in m future samples over the largest result in n previous samples is

$$X_m=\frac{m}{n+1}$$

The majority of the results to be presented illustrate the worst case in 100 samples, so that the mean number of exceedances in the next 1 sample is 1/101, or about 1 percent. Thus, the worst case in 100 samples represents about 99-percent probability of not being exceeded. This is felt to be a valid preliminary answer.

RESULTS AND DISCUSSION

The effect of data-adjustment and logic techniques on the performance of guidance maneuvers is now evaluated. In order best to describe the characteristics of the results, a reference case is presented first. A typical planet-to-planet transfer is chosen, and representative initial and measurement errors are assumed. The reference case used represents the result of a restricted optimization of guidance logic. Following this, the parametric results leading to the optimum choice are shown, and the effects of guidance logic are discussed. The remaining considerations of the statistical performance evaluation include the effect of variations in initial and measurement errors assumed.

REFERENCE CASE

Assumed parameters.—The mission under consideration is a 192-day one-way transfer from Earth to Mars with a launch date of December 13, 1964. The descriptive parameters of the reference trajectory are listed in table I along with the nominal values of errors and guidance-logic factors. The vehicle departs from Earth with a relative velocity (hyperbolic velocity) of about 13,000 feet per second, traverses an elliptical orbit inclined ½° to the ecliptic plane, and reaches a rendezvous with Mars at a relative velocity of about 18,000 feet per second.

The second group of assumed values includes the initial and measurement errors. The heliocentric injection velocity, which in this case is about 110,000 feet per second, corresponding to a launch velocity of 40,000 feet per second, is assumed to have a 70-foot-per-second-rms error in each of its three components. This error results in a position deviation at arrival of about 400,000 miles rms. The optical device is accurate to 10 seconds of arc, and the clock to 0.001 percent of the accumulated time. These instrument errors have been suggested in the literature as representative (ref. 9), although a valid question may arise as to whether 10 seconds of arc can actually be obtained from an integrated guidance system, the optical device being just one component of the system. The errors in applying Δv are taken as $\frac{1}{10}$ percent in magnitude and 20 seconds of arc in direction, which is twice the basic angular error.

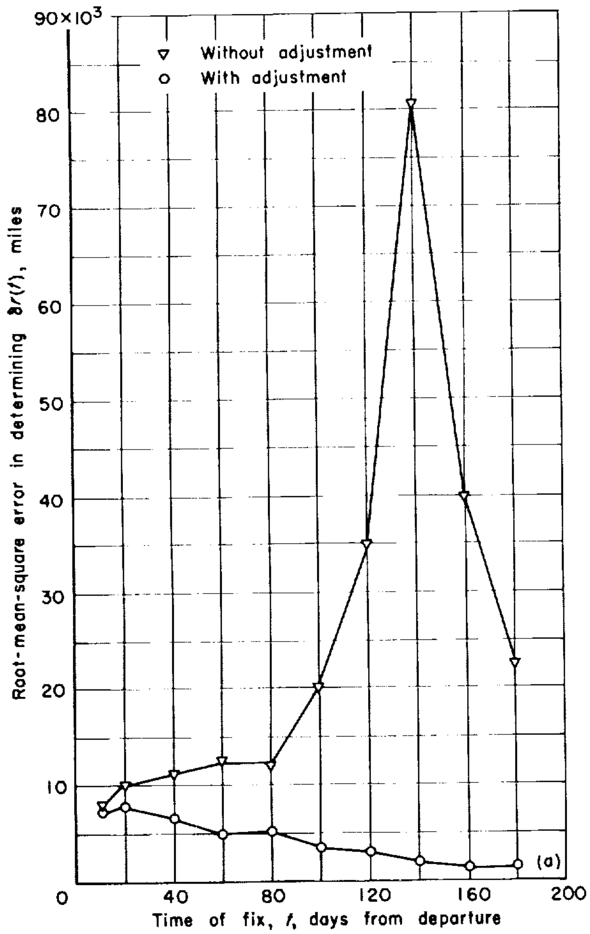
The third group of assumed values are the guidance-logic factors. The dead-band and damping coefficients are both 0.5. The minimum efficient velocity increment is taken as 10 feet per second.

Navigation fix.—The nature of the data-adjustment technique is that knowledge improves with the number of fixes taken. However, in an analysis of this type computing time is an important factor, and the number of fixes must be limited. (Note: Computing time may not be a restriction on board the vehicle, since the Monte Carlo method is not needed; however, increased storage capacity would be a consideration.)

Ten navagation fixes are taken, one at 10 days after departure, the second at 20 days, and the remainder at 20-day intervals thereafter. Stars 1 and 2 are chosen to have directions near the y and z axes of the ecliptic coordinate system. Mars and Earth are used as the planets to be sighted, with the constraint that the closest of the two be used for a given fix unless the Sun-planet angle is less than 15°, which is defined as a "nonvisible" condition. The descriptive parameters of the navigation sighting are listed in the following table:

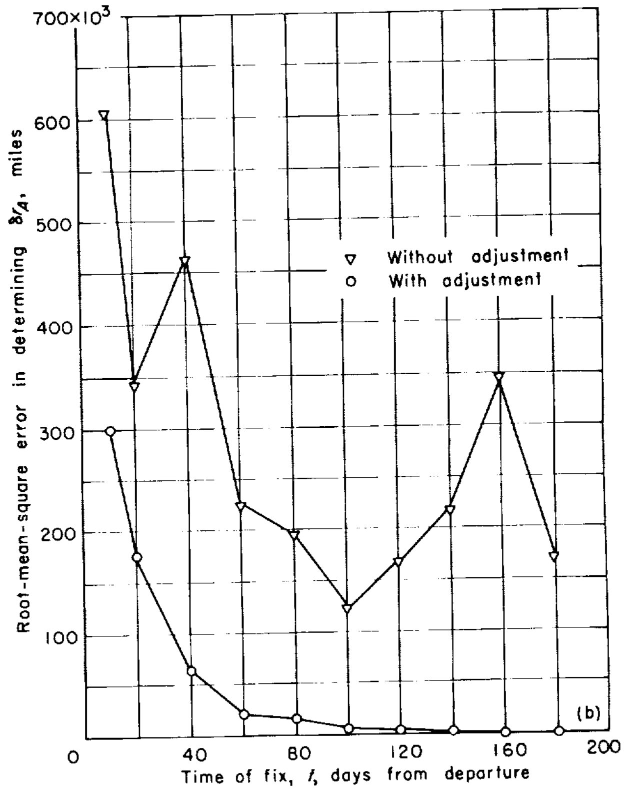
Time of fix, days	Planet sighted	Distance from planet sighted, miles	Reference angles, deg		
			Sun-star 1	Sun-star 2	Sun-planet
10	Earth	2.13×10^6	177	90.1	42.6
20	Earth	4.30	166	90.2	30.8
40	Mars	76.2	147	90.4	121
60	Mars	59.7	129	90.5	134
80	Earth	25.9	114	90.5	25.9
100	Mars	34.1	100	90.5	156
120	Mars	24.6	88.5	90.6	167
140	Mars	16.7	77.7	90.5	175
160	Mars	9.90	67.7	90.5	169
180	Mars	3.68	58.5	90.4	160

Effect of data adjustment.—To illustrate the effects of data adjustment on the accuracy of guidance knowledge, the rms error in determining the magnitude of position deviation $\delta r(t)$ is plotted against the time of fix in figure 5(a). The rms error is obtained from the square root of the sum of the diagonal elements of Σ_3 (eq. (B12)) The upper characteristic represents the error without any adjustment. The error at 10 days is about 8000 miles, increases to 80,000 miles at 140 days, and then decreases to 22,000 miles at the last fix. In contrast, the error characteristic for data adjustment shows a rather steady decline from 7000 to 1700 miles. The greatest



(a) Error characteristic for position deviation at time t.

FIGURE 5.—Results of reference case. Assumed parameters, table I.



(b) Error characteristic for position deviation at arrival.

FIGURE 5.—Continued.

increase in knowledge is about 40-fold, occurring at 140 days.

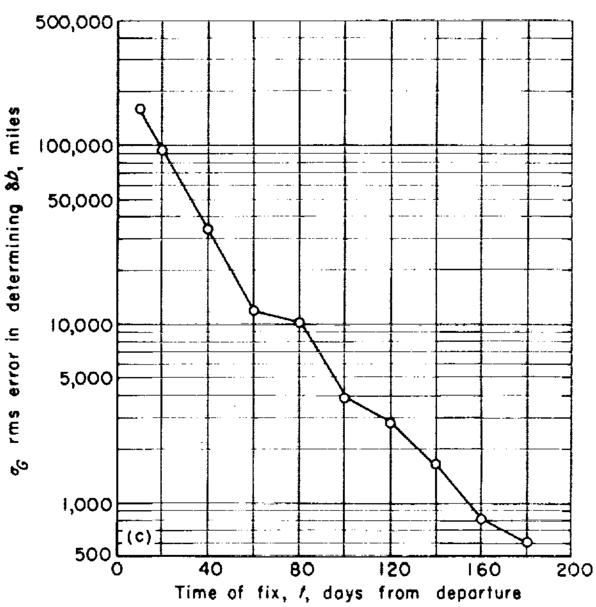
The effect of adjustment upon the accuracy of predicting the position deviation at arrival δr_A is even more evident (fig. 5(b)). Here, the rms error is computed from the matrix Σ_1. The error characteristic without adjustment is rather oscillatory, with a maximum error of 600,000 miles at 10 days, and a minimum error of 125,000 miles at 100 days. When adjustment is included, the error characteristic declines rapidly from 300,000 miles at 10 days to 21,000 miles at 60 days, and thereafter decreases to about 1800 miles at the final fix. The greatest increase in accuracy is about 170-fold for the fix taken 160 days after departure.

The advantage of the data-adjustment technique has been clearly indicated by figures 5(a) and (b) in terms of rms errors. It is felt both necessary and interesting, however, to study the details of data adjustment for the simulated random guidance maneuver. The parameter studied is $\delta \bar{r}_A$, and the following table compares the actual, uncorrected, and corrected values for five random samples taken from the data at $t=100$ days.

Position deviation at arrival, $\delta \bar{r}_A$	x-component, miles	y-component, miles	z-component, miles
Actual	3, 550	2, 094	551
Predicted before adjustment	145, 000	−16, 100	11, 600
Predicted after adjustment	−5, 210	4, 920	1, 940
Actual	6, 020	3, 140	357
Predicted before adjustment	−139, 000	−16, 400	8, 850
Predicted after adjustment	10, 400	1, 480	892
Actual	−15, 600	9, 970	1, 490
Predicted before adjustment	65, 700	11, 600	24, 800
Predicted after adjustment	−12, 300	8, 870	−2, 050
Actual	−21, 800	10, 000	1, 030
Predicted before adjustment	76, 800	34, 400	−19, 800
Predicted after adjustment	−19, 200	8, 760	355
Actual	12, 500	4, 320	2, 760
Predicted before adjustment	−7, 800	3, 340	50, 100
Predicted after adjustment	13, 100	−2, 970	4, 690

A scan of this table shows that guidance knowledge is significantly improved because of the adjustment.

The error characteristic of the miss distance δb is plotted in figure 5(c) on semilog scales. Each



(c) Error characteristic for miss distance.

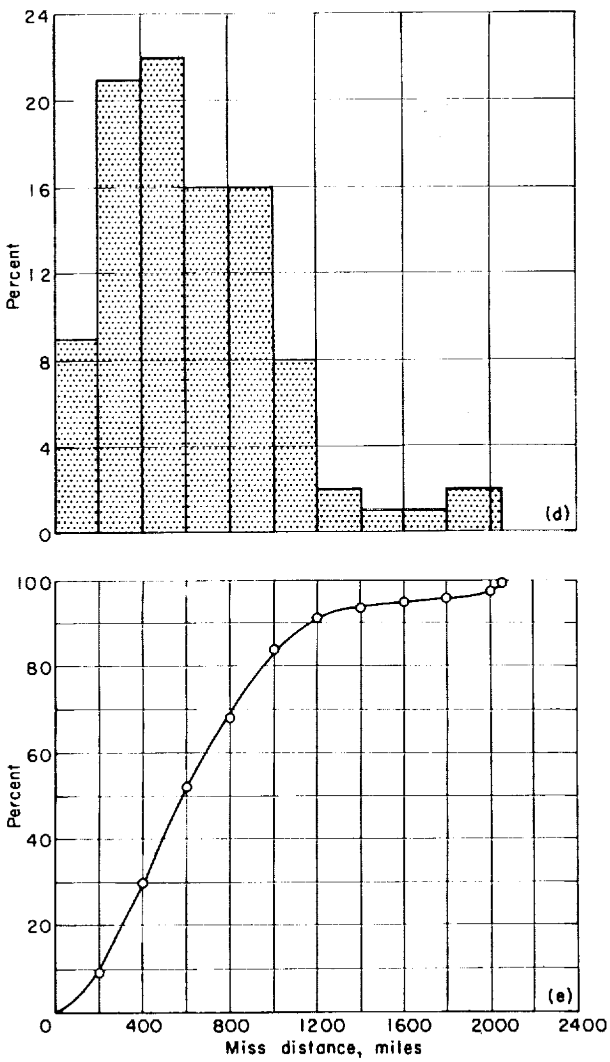
FIGURE 5. Continued.

point may be thought of as the final accuracy attainable if the last correction were made at that time. If only a single correction were made at 10 days after departure, the rms miss distance would be about 150,000 miles. The nine subsequent fixes ending at 180 days cause the miss to be reduced by more than two decades to 600 miles. Furthermore, results show that the increased Δv added by the subsequent corrections is rather insignificant compared to the increased accuracy.

Results of guidance maneuver. The final accuracy of the guidance maneuver is shown in figure 5(d). Here the statistical distribution of miss distance is plotted in the form of a rectangular frequency (density) polygon. The most probable range of miss distance, corresponding to the region of highest frequency, is between 200 and 1000 miles, with an average of about 600 miles. It is recalled that the rms error in predicting miss distance at the final guidance point was also on the order of 600 miles. The shape of the distribution is skewed considerably toward higher miss distances, this being characteristic of normally distributed errors.

The results of guidance accuracy may be presented in another form, namely, the integrated frequency polygon commonly called the cumulative probability distribution. Figure 5(e) shows the probability of not exceeding a given size miss distance. The median miss distance (50 percentile) is about 600 miles, and the probability of not exceeding 1200 miles is about 90 percent. The largest miss distance obtained from 100 random samples was 2050 miles and as previously discussed may be considered the 99-percent probability level.

The cost of guidance in terms of total velocity expenditure is shown in figure 5(f) in frequency-polygon form. The most probable Δv_t requirement is about 100 feet per second, but this distribution is also skewed toward higher total velocity increments. Figure 5(g) illustrates the cumulative probability distribution. The median

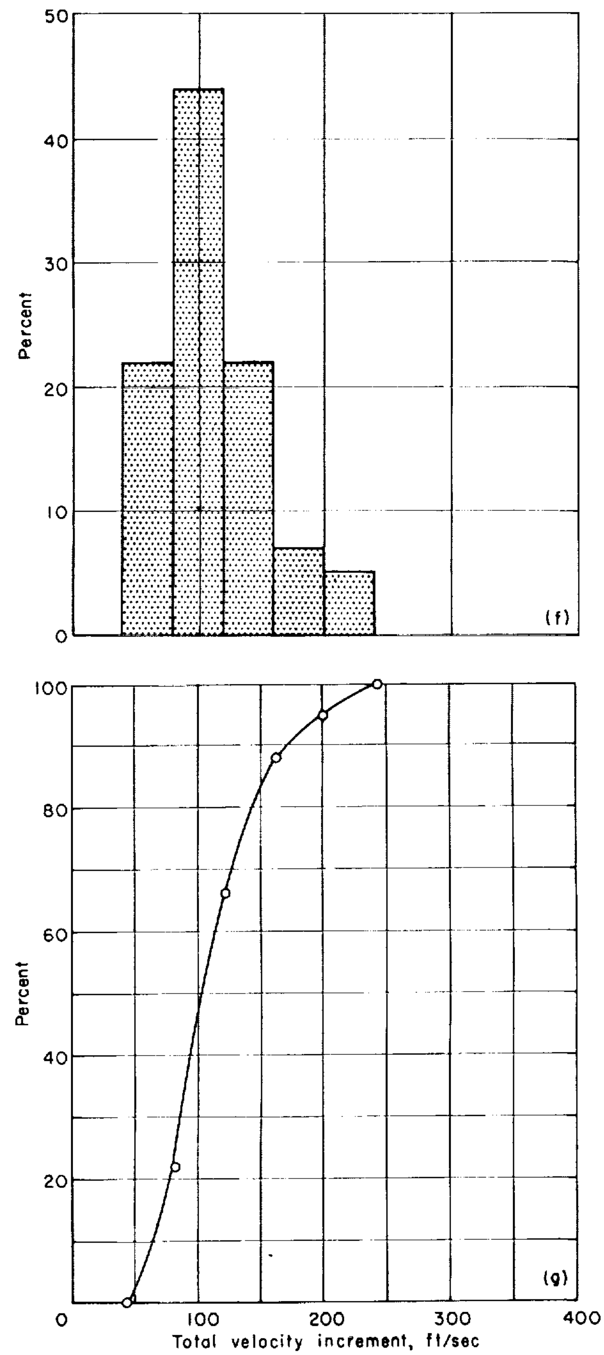


(d) Frequency distribution of target miss distance.
(e) Cumulative probability distribution of target miss distance.

FIGURE 5. Continued.

Δv_t is 100 feet per second, which in this case is about equal to the most probable requirement. The largest Δv_t obtained from 100 samples was 235 feet per second.

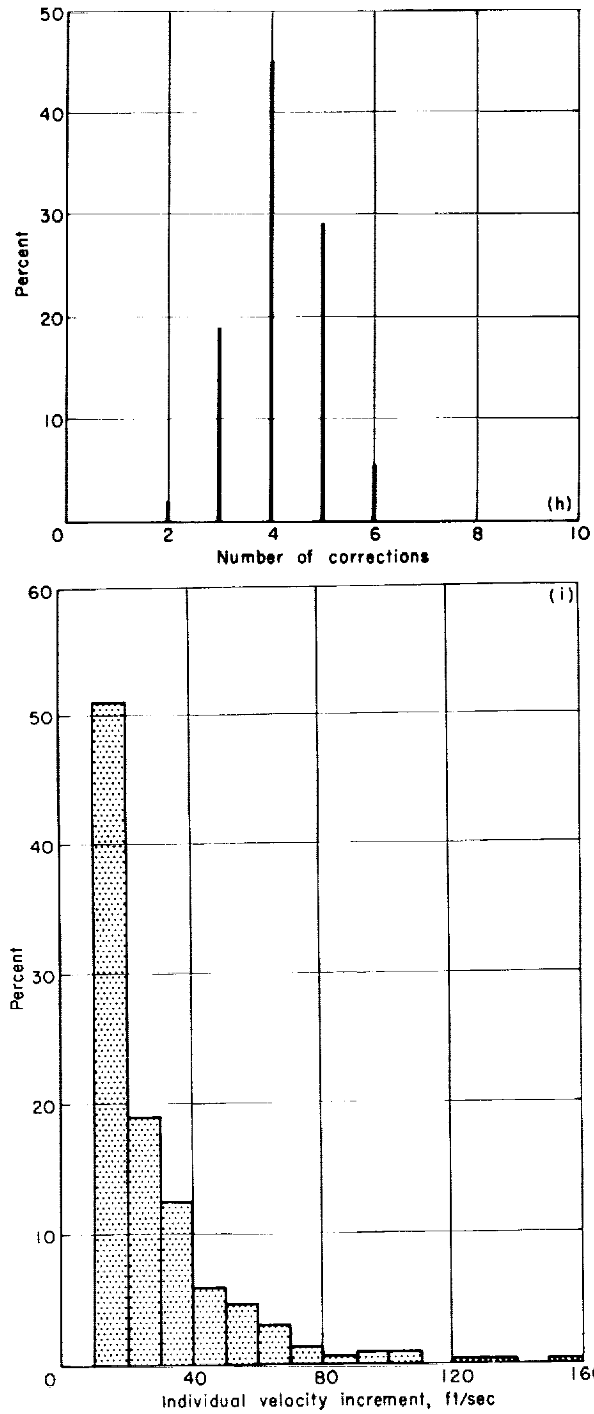
Figure 5(h) shows the probability of using a given number of corrections during the guidance maneuver. The number of corrections ranges from 2 to 6, with probabilities equal to 2, 19, 45, 29, and 5 percent. The effect of guidance logic, then, is to reject an average of 6 out of 10 possible corrections. Breaking this down further into the effects of dead band and Δv_{min}, it was found that about 80 percent of the corrections rejected



(f) Frequency distribution of total velocity increment.
(g) Cumulative probability distribution of total velocity increment.

FIGURE 5.—Continued.

were due to Δv_{min}. In this particular example, the minimum limit of velocity increment was the more effectiue in reducing the number of corrections.



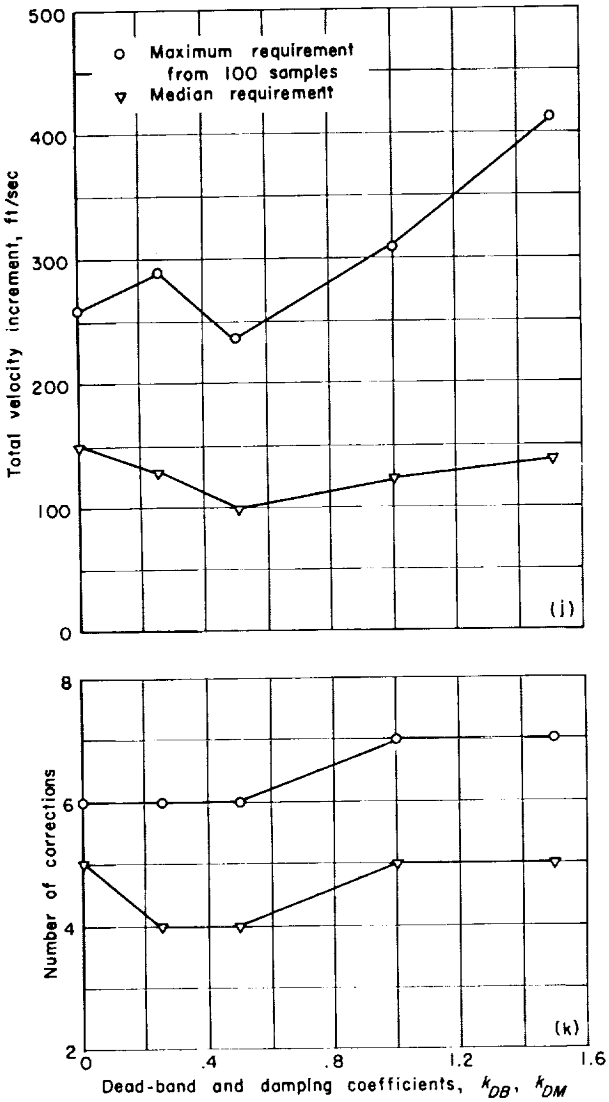
(h) Frequency distribution of number of corrections.
(i) Frequency distribution of individual velocity increment.

FIGURE 5.—Continued.

The range of Δv anticipated and the frequency of a given Δv increment may be of possible importance to engine design. The frequency distribution of individual velocity increments is shown in figure 5(i). Increments less than 10 feet per second are not used because of guidance logic. The largest Δv required is 160 feet per second, but increments of this order are very infrequent (¼ percent). Most frequently used Δv (51 percent) is between 10 and 20 feet per second.

Effect of dead band and damping.—The choice of dead-band and damping coefficients equal to 0.5 was arrived at by determining the effects of variations in these factors on the requirements of total velocity increment and number of corrections. The results presented in figure 5(j) show the effect of k_{DB} and k_{DM} on Δv_t. Although the maximum requirement is of first importance, the median requirement is included to verify trends. The maximum total velocity increment for no dead band or damping is 260 feet per second, and for k_{DB} and k_{DM} equal to 0.5 it is 235 feet per second. The velocity requirement increases with coefficients above 0.5; 415 feet per second is needed for k_{DB} and k_{DM} equal to 1.5. The explanation of this characteristic lies in the fact that a trade-off exists between the amount of wasted Δv (in a statistical sense) at a given correction point when low damping is used and the amount of Δv required at later points to correct the significant trajectory error that remains when high damping is used. It is recognized that the cost of correcting a given size error increases as the target is approached. The results of this study indicate that relatively small dead band and damping, about 0.5, should be used to minimize Δv_t. This conclusion was also reached in a study of approach guidance (ref. 6). It is noted, however, that the difference in Δv required for k_{DB} and k_{DM} equal to 0 and 0.5 is rather insignificant, about 25 feet per second. This difference becomes more important when larger measurement errors are assumed and will be indicated later.

The effect of guidance logic on the number of corrections required is shown in figure 5(k). The maximum number of corrections is 6 for k_{DB} and k_{DM} equal to 0, 0.25, and 0.5. The maximum number is 7 for k_{DB} and k_{DM} equal to 1.0 and 1.5; this increase is due to the less significant dead-band effect of Δv_{min} when higher damping is used. The number of corrections should decrease again for very high values of k_{DB} and k_{DM}, since



(j) Effect of dead band and damping on velocity increment.
(k) Effect of dead band and damping on number of corrections.

FIGURE 5. Concluded.

most corrections will be rejected by the insignificant knowledge criterion. These cases are of no interest, however, because of the excessive Δv_t that would be required.

EFFECT OF ERROR ASSUMPTIONS

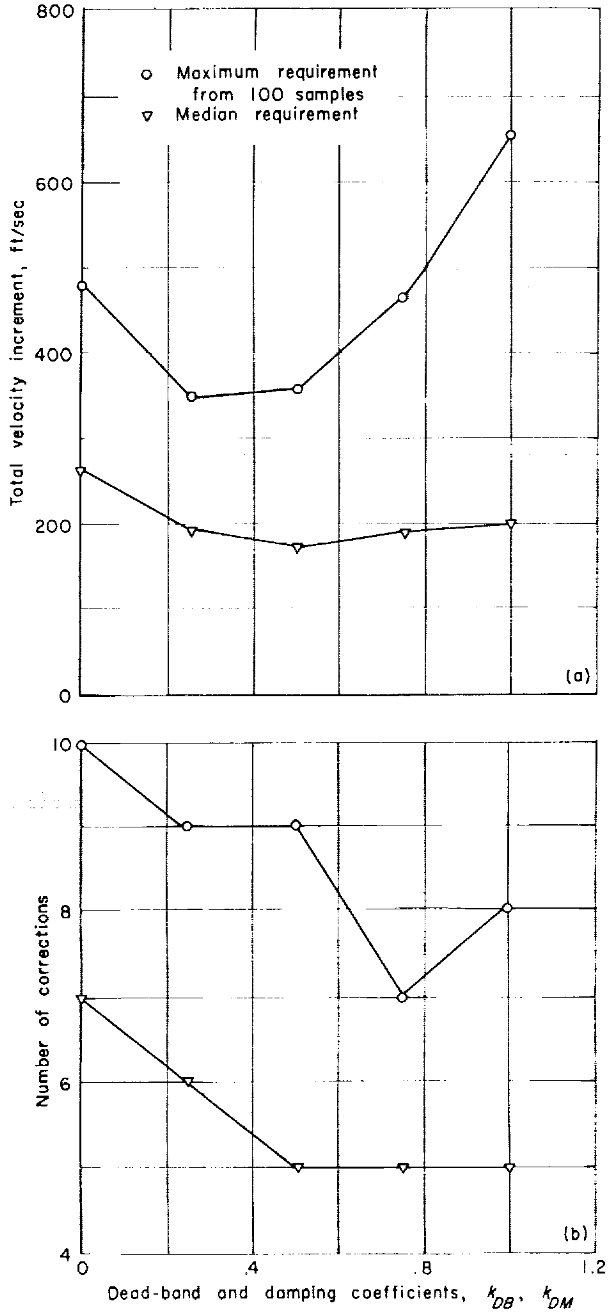
The following sections present the results of a parametric variation of initial errors and measurement errors as they affect guidance performance. All parameters not specifically varied are those of the reference case as summarized in table I.

Increased sighting error.—The accuracy and Δv cost of guidance are affected to a large extent by the size of the measurement error distributions. Clocking errors have been found to contribute a rather small effect; consequently, they are excluded from the following discussion. Guidance performance was evaluated for an increased sighting error of 40 and 120 seconds of arc rms, and the major results are presented in figures 6 and 7.

Figure 6(a) shows the effect of dead-band and damping guidance logic on the total velocity-increment requirement. The general characteristic shown in figure 5(j) for 10 seconds of arc error is repeated herein; however, the effect of guidance logic on reducing Δv_t is more significant. The optimum choice of k_{DB} and k_{DM} again appears to be about 0.5, with a maximum Δv_t requirement on the order of 355 feet per second. Since the maximum Δv_t for k_{DB} and k_{DM} equal to zero is 480 feet per second, optimum logic reduces the cost of guidance by 25 percent. The accuracy of guidance is not shown, since it is unaffected by dead band and damping. Results indicate a median miss distance of 1200 miles and a maximum miss distance of 4500 miles. One might have expected the miss distance to be 4 times that for the 10-second-of-arc measurement error; however, the effect of data adjustment is to increase the miss distance by a factor of 2 only.

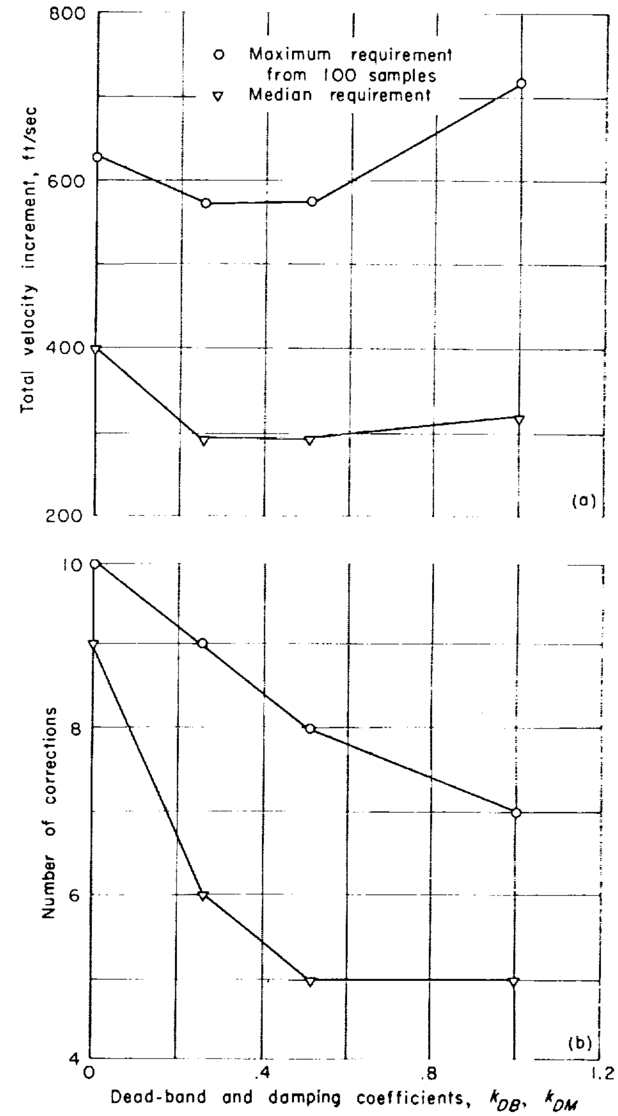
Figure 6(b) shows the effect of dead band and damping on the number of corrections required. When k_{DB} and k_{DM} equal zero, the maximum and average number of corrections are 10 and 7, respectively. When k_{DB} and k_{DM} equal 0.5, the maximum and average are 9 and 5, respectively. The fact that Δv_{min} is a less effective dead band for 40 seconds of arc than for 10 seconds of arc causes the number of corrections to increase.

The effect of dead band and damping for a sighting error of 120 seconds arc is illustrated in figure 7. A value of k_{DB} and k_{DM} equal to 0.5 again appears to be a suitable choice. The maximum and median Δv requirements are about 575 and 300 feet per second, respectively; and the maximum and average number of corrections are 8 and 5, respectively. Although not shown, results indicate a maximum miss distance of about 12,600 miles and a median miss distance of 4000 miles.



(a) Total velocity increment.
(b) Number of corrections.

FIGURE 6.—Effect of dead band and damping. Root-mean-square sighting error, 40-second arc; Δv direction error, 80-second arc.



(a) Total velocity increment.
(b) Number of corrections.

FIGURE 7.—Effect of dead band and damping. Root-mean-square sighting error, 120-second arc; root-mean-square Δv direction error, 240-second arc.

Increased injection-velocity error.—The reference case presented the results for an assumed injection-velocity error of 70 feet per second rms in each component. This represented, essentially, a $\frac{1}{10}$-percent error in the burnout velocity at launch. In figure 8, the total Δv requirement is plotted as a function of the injection-velocity error for three values of sighting error. In the case of 10-second-arc sighting error, the characteristic is approximately linear; the maximum Δv requirement increases from 160 feet per second for a 40-foot-per-second injection error to 520 feet per second for a 200-foot-per-second injection error. When sighting errors of 40 and 120 seconds are are considered, the maximum Δv requirement increases from 260 to 730 feet per second and from 400 to 950 feet per second, respectively.

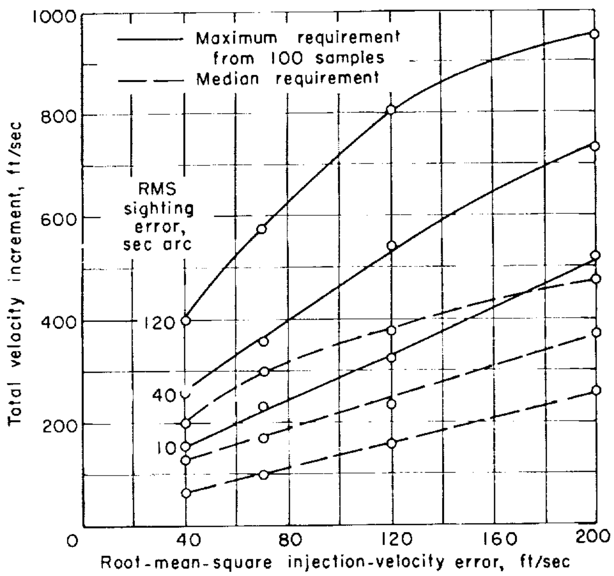


FIGURE 8. Effect of injection-velocity error on Δv requirements.

CONCLUDING REMARKS

The major objective of this report was to formulate and evaluate a guidance theory that may be employed to prescribe efficient trajectory control during the midcourse phase of an interplanetary mission. The basic guidance theory, which has been given attention in the literature (ref. 9), was developed by perturbation methods considering first-order deviations from a precomputed reference trajectory. A self-contained optical measurement system was hypothesized, consisting of tracking devices to measure angles between pairs of celestial bodies, a clock, and instrumentation for thrust vector control. Errors in all measurements were considered and specified by statistical distributions. Corrective maneuvers were so applied as to cause the vehicle to reach rendezvous with the target planet at approximately the reference time of arrival. Furthermore, for the purpose of this analysis, the target parameter was specified only in terms of the asymptotic miss distance.

The criterion of an acceptable guidance theory is one that will guide the vehicle to a reasonable degree of final accuracy and in so doing expend a minimum amount of propellant. In addition, it is desirable to hold down the number of corrections required and to relax the error tolerance of practical instrumentation. The method adopted in this analysis to improve guidance efficiency was twofold:

(1) A technique of data adjustment was included in the theory with the purpose of increasing the accuracy of guidance information. This was accomplished by including knowledge of the past history of the vehicle trajectory along with current guidance information and reducing the redundant information by a "maximum-likelihood" adjustment. The nature of this adjustment procedure was improved guidance information as the number of navigation fixes increases. A numerical evaluation indicated typical results of improved over nonadjusted data accuracy anywhere from 2- to 170-fold. This improved accuracy was responsible for a significant reduction of Δv cost.

(2) So-called guidance-logic or decision expressions were included in the theory and reduced the number of unnecessary and wasteful corrections inherent to guidance maneuvers in the face of random measurement errors. The predicted miss distance at the target and its estimated variance are interpreted by the logic, and a decision is made as to whether a correction is warranted, and if so, what amount of Δv is to be applied. The logic is based on arbitrary dead-band and damping coefficients, which were varied parametrically to result in optimum performance. Results of this study indicated that small dead band and damping were desirable. Evaluation of guidance using 40 seconds of arc rms measurement accuracy showed that the logic was effective in rejecting an average of 5 out of 10 possible corrections with a corresponding 25-percent reduction of Δv.

Guidance performance was evaluated by simulating the guidance maneuver on a digital computer and using Monte Carlo techniques of statistical analysis. The mission considered is a 192-day Earth-Mars transfer. An injection-velocity error of 70 feet per second rms is assumed, which would cause a miss distance of several hundred thousand miles. Ten navigation fixes are taken en route. With tracking errors of 10 seconds arc rms, the largest total Δv requirement obtained from 100 samples was about 235 feet per second, and the largest miss distance incurred was about 2000 miles. The maximum number of corrections required was six. With increased tracking errors of 40 seconds arc and 2 minutes arc rms, the Δv requirement increased to 355 and 575 feet per second, respectively, while the miss distance increased to 4500 and 12,600 miles, respectively. In these cases, the maximum number of corrections increased to nine and eight, respectively.

LEWIS RESEARCH CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CLEVELAND, OHIO, *May 18, 1961*

APPENDIX A

DEVELOPMENT OF FUNDAMENTAL GUIDANCE THEORY

The literature contains numerous references (e.g., 8 and 9) to the development of a linearized guidance theory. The basic idea in this approach is to define a standard or reference trajectory along which the space vehicle will move in the ideal situation. The assumption is then made that ensuing perturbations (e.g., guidance errors) are not great enough to cause the actual trajectory to deviate significantly from the reference trajectory. It is then possible to study these deviations and the required correctional maneuvers by perturbation techniques. The following development is to a large extent similar to the comprehensive analysis of reference 9. Vector and matrix notation is used throughout for the purpose of compactness and ease in algebraic manipulation.

Let $\bar{r}(t)$ and $\bar{v}(t)$ denote the actual position and velocity vectors at time t given in an inertial coordinate system centered at the Sun. Similarly, let $\bar{r}_o(t)$ and $\bar{v}_o(t)$ denote these quantities along the reference trajectory. The respective deviations are written as

$$\delta\bar{r}(t)=\bar{r}(t)-\bar{r}_o(t) \tag{A1}$$

$$\delta\bar{v}(t)=\bar{v}(t)-\bar{v}_o(t) \tag{A2}$$

The basic differential equations describing coasting motion in a gravitational field are

$$\frac{d\bar{r}}{dt}=\bar{v} \tag{A3}$$

$$\frac{d\bar{v}}{dt}=\bar{g}(\bar{r},t) \tag{A4}$$

where $\bar{g}$ is the acceleration due to the field.

Differentiating (A1) and (A2) with time, and using (A3) and (A4),

$$\frac{d}{dt}\,\delta\bar{r}(t)=\bar{v}(t)-\bar{v}_o(t)$$

$$\frac{d}{dt}\,\delta\bar{v}(t)=\bar{g}(\bar{r},t)-\bar{g}(\bar{r}_o,t)$$

$\bar{g}(\bar{r},t)$ may be expanded in terms of a Taylor series about the point $\bar{r}_o$; and, since deviations have been assumed small, all higher-order terms in $(\bar{r}-\bar{r}_o)$ may be neglected, with the result

$$\bar{g}(\bar{r},t)-\bar{g}(\bar{r}_0,t)=M_0(\bar{r}_0,t)\delta\bar{r}(t)$$

Here M_o is a coefficient matrix whose elements are the partial derivatives of the components of $\bar{g}(\bar{r}_o,t)$ with respect to the components of $\bar{r}_o(t)$. Specifically,

$$M_0=\begin{bmatrix}\frac{\partial g_x}{\partial r_x} & \frac{\partial g_x}{\partial r_y} & \frac{\partial g_x}{\partial r_z}\\[6pt] \frac{\partial g_y}{\partial r_x} & \frac{\partial g_y}{\partial r_y} & \frac{\partial g_y}{\partial r_z}\\[6pt] \frac{\partial g_z}{\partial r_x} & \frac{\partial g_z}{\partial r_y} & \frac{\partial g_z}{\partial r_z}\end{bmatrix}_{\bar{r}_0,t} \tag{A5}$$

The linearized differential equations of perturbed motion may now be written

$$\frac{d}{dt}\,\delta\bar{r}(t)=\delta\bar{v}(t) \tag{A6}$$

$$\frac{d}{dt}\,\delta\bar{v}(t)=M_o\delta\bar{r}(t) \tag{A7}$$

The general solution of equations (A6) and (A7) that meets certain specified boundary conditions is desired. An appropriate set of boundary conditions may be

$$\delta\bar{r}(t_L)=0; \qquad \delta\bar{v}(t_L)=\delta\bar{v}_L \tag{A8}$$

$$\delta\bar{r}(t_A)=0; \qquad \delta\bar{v}(t_A)=\delta\bar{v}_A \tag{A9}$$

Equation (A8) states that the initial perturbation is due solely to errors in velocity at launch. Equation (A9) states that any position deviation at arrival is to be nulled by the guidance maneuver; however, the velocity deviation at arrival will not be constrained.

Assume that the vehicle has been coasting since time of departure t_L, and that at time t the position and velocity vectors are $\bar{r}(t)$ and $\bar{v}(t)$. These quantities are functions solely of departure velocity $\bar{v}(t_L)$. A Taylor series expansion about the reference value $\bar{v}_o(t_L)$ with higher-order terms neglected yields

$$\delta\bar{r}(t)=R(t)\delta\bar{v}_L \tag{A10}$$

$$\delta\bar{v}(t)=V(t)\delta\bar{v}_L \tag{A11}$$

where $R(t)$ and $V(t)$ are coefficient matrices of partial derivatives evaluated along the reference trajectory at time t. Defining ∇ as a row vector,

$$\nabla=\left(\frac{\partial}{\partial v_{L,x}}\frac{\partial}{\partial v_{L,y}}\frac{\partial}{\partial v_{L,z}}\right) \tag{A12}$$

$$R(t)=\begin{bmatrix}\nabla r_x(t)\\ \nabla r_y(t)\\ \nabla r_z(t)\end{bmatrix};\quad V(t)=\begin{bmatrix}\nabla v_x(t)\\ \nabla v_y(t)\\ \nabla v_z(t)\end{bmatrix} \tag{A13}$$

Substituting (A10) and (A11) into (A6) and (A7),

$$\frac{dR}{dt}\delta\bar{v}_L=V\delta\bar{v}_L$$

$$\frac{dV}{dt}\delta\bar{v}_L=M_oR\delta\bar{v}_L$$

Thus,

$$\frac{dR}{dt}=V;\quad \frac{dV}{dt}=M_oR \tag{A14}$$

Equations (A10) and (A11) form one solution to the perturbed differential equations (A6) and (A7) integrating forward from t_L. Furthermore, from the boundary conditions of (A8),

$$R(t_L)=0;\quad V(t_L)=I(\text{unit matrix}) \tag{A15}$$

In an analogous manner one may picture coasting motion from the target planet at t_A backwards along the trajectory to time t. The position and velocity vectors at t are solely functions of arrival velocity $\bar{v}(t_A)$; consequently, the deviations are written

$$\delta\bar{r}(t)=R^*(t)\delta\bar{v}_A \tag{A16}$$

$$\delta\bar{v}(t)=V^*(t)\delta\bar{v}_A \tag{A17}$$

In this case the row vector ∇^* is defined

$$\nabla^*=\left(\frac{\partial}{\partial v_{A,x}}\frac{\partial}{\partial v_{A,y}}\frac{\partial}{\partial v_{A,z}}\right) \tag{A18}$$

and

$$R^*(t)=\begin{bmatrix}\nabla^* r_x(t)\\ \nabla^* r_y(t)\\ \nabla^* r_z(t)\end{bmatrix};\quad V^*(t)=\begin{bmatrix}\nabla^* v_x(t)\\ \nabla^* v_y(t)\\ \nabla^* v_z(t)\end{bmatrix} \tag{A19}$$

Also, as in (A14)

$$\frac{dR^*}{dt}=V^*;\quad \frac{dV^*}{dt}=M_oR^* \tag{A20}$$

Equations (A16) and (A17) form another solution to (A6) and (A7) integrating backwards from t_A. The boundary conditions of (A9) then require

$$R^*(t_A)=0;\quad V^*(t_A)=I \tag{A21}$$

The most general solution to the perturbed differential equations is thus

$$\delta\bar{r}(t)=R(t)\bar{c}+R^*(t)\bar{c}^* \tag{A22}$$

$$\delta\bar{v}(t)=V(t)\bar{c}+V^*(t)\bar{c}^* \tag{A23}$$

where $\bar{c}$ and $\bar{c}^*$ are constant vectors over a coasting-time interval and may be determined from any set of boundary conditions.

It should be noted that the preceding development applies to any number of gravitational fields acting simultaneously and requires only that the matrices R, V, R^*, and V^* be obtained. It is possible to obtain these quantities from the simultaneous solution of equations (A14) and (A20), and the differential equations of motion along the reference trajectory. A total of 48 differential equations would be required. However, in the special case of a predominant solar field, the matrices can be obtained analytically from the equation of two-body Keplerian motion. It is not expected that a more accurate definition would significantly affect the results to be developed in the present study.

The next step is to derive the basic equations governing correctional maneuvers. It will be shown that the velocity increment required to null any position deviation at arrival is simply a linear function of two known position deviations along the trajectory. The following development is for "fixed-time-of-arrival guidance"; that is, correcting the trajectory to intercept the target planet at the reference time of arrival t_A. The theory may easily be extended to "variable-time-of-arrival guidance" for the purpose of minimizing the required velocity increment.

Let $\bar{v}^*(t)$ represent the velocity vector required at $\bar{r}(t)$ to intercept the moving target at t_A. Expanding $\bar{v}^*(t)$ in a Taylor series about the reference trajectory and neglecting all terms in $\delta\bar{r}(t)$ of higher order than one,

$$\bar{v}^*(t)=\bar{v}_o(t)+Q^*(\bar{r}_o,t)\delta\bar{r}(t) \tag{A24}$$

where Q^* is the matrix of partial derivatives of the components of $\bar{v}^*(t)$ with respect to $\bar{r}(t)$ evaluated at $\bar{r}_0(t)$. The required velocity correction is then

$$\Delta\bar{v}(t)=\bar{v}^*(t)-\bar{v}(t)=Q^*\delta\bar{r}(t)-\delta\bar{v}(t) \qquad \text{(A25)}$$

The quantity $\delta\bar{v}(t)$ is not directly measurable with the sensing equipment assumed, but it may be found from two successive position deviations. Equation (A22) written for times t_n and t_{n-1} will yield $\bar{c}$ and $\bar{c}^*$, which can then be substituted into (A23). Denoting time-dependent quantities with subscripts, for example $\delta\bar{r}(t_n)\equiv\delta\bar{r}_n$,

$$\bar{c}=(R_n-R_n^*R_{n-1}^{*-1}R_{n-1})^{-1}(\delta\bar{r}_n-R_n^*R_{n-1}^{*-1}\delta\bar{r}_{n-1})$$

$$\bar{c}^*=(R_n^*-R_nR_{n-1}^{-1}R_{n-1}^*)^{-1}(\delta\bar{r}_n-R_nR_{n-1}^{-1}\delta\bar{r}_{n-1})$$

From the definition of Q^* and equations (A13), it is seen that

$$Q_n^*R_n^*=V_n^*$$

These results substituted into (A25) give

$$\Delta\bar{v}_n=H_n\delta\bar{r}_n-P_n\delta\bar{r}_{n-1} \qquad \text{(A26)}$$

where H_n and P_n are matrices related to the fundamental matrices R, V, R^*, and V^*. Specifically,

$$H_n=V_n^*R_n^{*-1}-V_n(R_n-R_n^*R_{n-1}^{*-1}R_{n-1})^{-1} \\ -V_n^*(R_n^*-R_nR_{n-1}^{-1}R_{n-1}^*)^{-1} \qquad \text{(A27)}$$

$$P_n=-V_n(R_n-R_n^*R_{n-1}^{*-1}R_{n-1})^{-1}R_n^*R_{n-1}^{*-1} \\ -V_n^*(R_n^*-R_nR_{n-1}^{-1}R_{n-1}^*)^{-1}R_nR_{n-1}^{-1} \qquad \text{(A28)}$$

Equation (A26) expresses the velocity correction as a linear function of two measured position deviations. It is necessary to carry the analysis one step further and derive an expression for $\Delta\bar{v}_n$ that causes a position deviation at arrival equal to $\delta\bar{r}_A'$, not necessarily zero.

From (A22), $\delta\bar{r}_A'=R_A\bar{c}$, since $R_A^*=0$. Now $\delta\bar{r}_A'$ is desired after the nth velocity correction is applied; thus $\bar{c}$ can be found from the position and velocity deviations immediately after the correction. Position is essentially constant during the correction, and the velocity deviation is $(\Delta\bar{v}_n+\delta\bar{v}_n)$. The solution for $\bar{c}$ from equations (A22) and (A23) at time t_n is

$$\bar{c}=(V_n-V_n^*R_n^{*-1}R_n)^{-1}(\delta\bar{v}_n+\Delta\bar{v}_n-V_n^*R_n^{*-1}\delta\bar{r}_n) \qquad \text{(A29)}$$

Now, $\delta\bar{v}_n$, which in this case is the velocity deviation before the correction, depends upon the present and past position deviations:

$$\delta\bar{v}_n=(V_n^*R_n^{*-1}-H_n)\delta\bar{r}_n+P_n\delta\bar{r}_{n-1} \qquad \text{(A30)}$$

Denoting

$$B_n=R_A(V_n-V_n^*R_n^{*-1}R_n)^{-1} \qquad \text{(A31)}$$

$\delta\bar{r}_A'$ is found from equations (A29), (A30), and (A31):

$$\delta\bar{r}_A'=B_n(P_n\delta\bar{r}_{n-1}-H_n\delta\bar{r}_n+\Delta\bar{v}_n) \qquad \text{(A32)}$$

Solving for $\Delta\bar{v}_n$,

$$\Delta\bar{v}_n=B_n^{-1}\delta\bar{r}_A'+H_n\delta\bar{r}_n-P_n\delta\bar{r}_{n-1} \qquad (2)$$

It is noted that (2) reduces to (A26) when the correction nulls the deviation; that is, $\delta\bar{r}_A'=0$. Furthermore, (A32) also gives the position deviation at arrival before $\Delta\bar{v}_n$ is applied:

$$\delta\bar{r}_A=(B_nP_n)\delta\bar{r}_{n-1}-(B_nH_n)\delta\bar{r}_n \qquad (1)$$

APPENDIX B

DEVELOPMENT OF DATA-ADJUSTMENT TECHNIQUE

The general problem of least-squares adjustment of a set of observations subject to statistically distributed random errors has been treated comprehensively by Brown (ref. 11). The following development shows the application of the method to the midcourse guidance problem.

The basic condition equation (1) expresses the predicted position deviation at arrival $\delta\bar{r}_A$ as a linear function of two indirectly measured position deviations along the path:

$$\delta\bar{r}_A-(B_nP_n)\delta\bar{r}_{n-1}+(B_nH_n)\delta\bar{r}_n=0 \qquad \text{(B1)}$$

Current data acquisition is at time t_n. Since the quantities $\delta\bar{r}_{n-1}$ and δr_n are made up in part of random measurement errors, the prediction of $\delta\bar{r}_A$ will also be in error and will consequently affect any guidance action attempted. What is desired then is a means of improving the accuracy of the prediction.

Now suppose that at time t_{n-1}, which is the previous data-acquisition and guidance point, a prediction of $\delta\bar{r}_A$ and a measure of the accuracy in that prediction are available. Since the vehicle has followed a coasting trajectory between t_{n-1} and t_n, this prediction is valid for current use at t_n. The $\delta\bar{r}_A$ may be treated as an indirectly measured quantity in (B1), although strictly speaking it is a parameter. The following adjustment procedure will act on the redundant guidance information in such a manner as to yield a "best" estimate of the quantities $\delta\bar{r}_A$, $\delta\bar{r}_{n-1}$, and $\delta\bar{r}_n$.

Denoting $\delta\bar{r}_A^\circ$, $\delta\bar{r}_{n-1}^\circ$, and $\delta\bar{r}_n^\circ$ as the indirectly measured quantities, the adjusted quantities are written as

$$\left.\begin{aligned}\delta\bar{r}_A&=\delta\bar{r}_A^\circ+\bar{\gamma}_1\\ \delta\bar{r}_{n-1}&=\delta\bar{r}_{n-1}^\circ+\bar{\gamma}_2\\ \delta\bar{r}_n&=\delta\bar{r}_n^\circ+\bar{\gamma}_3\end{aligned}\right\} \qquad \text{(B2)}$$

where the $\bar{\gamma}$ are at present undetermined corrections called residuals. At the measured conditions, the right side of (B1) is not equal to zero in general and may be represented by a discrepancy vector $\bar{\mathscr{E}}$, where

$$\bar{\mathscr{E}}=\delta\bar{r}_A^\circ-(B_nP_n)\delta\bar{r}_{n-1}^\circ+(B_nH_n)\delta\bar{r}_n^\circ \qquad \text{(B3)}$$

Eliminating $\delta\bar{r}_A$, $\delta\bar{r}_{n-1}$, and $\delta\bar{r}_n$ through equations (B1), (B2), and (B3), the so-called normal condition equation is expressed in terms of the residual and discrepancy vectors. Using matrix notation,

$$A\bar{\gamma}+\bar{\mathscr{E}}=0 \qquad \text{(B4)}$$

where

$$A=\left[\,I\;\Big|\;-B_nP_n\;\Big|\;B_nH_n\,\right];\qquad \bar{\gamma}=\begin{Bmatrix}\bar{\gamma}_1\\ \bar{\gamma}_2\\ \bar{\gamma}_3\end{Bmatrix} \qquad \text{(B5)}$$

It is noted that (B4) when expanded results in three equations in nine unknown residuals. Therefore, additional criteria must be specified to obtain a solution. To arrive at such criteria, the statistical distribution of the errors is considered briefly.

The primary measurement errors ($\Delta\alpha_1$, $\Delta\alpha_2$, $\Delta\alpha_3$, Δt) are assumed mutually independent and normally distributed. Since the indirect measured quantities of interest are derived from the primary measurements, their errors have the statistical property of the general multivariate normal distribution. This result is found by Brown (ref. 11) and Cramer (ref. 12). The distribution of the derived errors (residuals) can be written

$$\varphi(\gamma_1,\gamma_2,\ .\ .\ .,\gamma_9)=\left(\frac{1}{2\pi}\right)^{9/2}(|\Sigma^{-1}|)^{1/2}e^{-\frac{1}{2}(\bar{\gamma}^T\Sigma^{-1}\bar{\gamma})} \qquad \text{(B6)}$$

where the matrix Σ is the covariance matrix of the derived measurements, and $|\Sigma^{-1}|$ is the determinant of its inverse matrix. Since the derived errors are not mutually independent, the covariance matrix is nondiagonal. Brown extends the work of earlier investigators and solves the general problem of "least-squares" adjustment considering correlated errors. Actually, he terms his method the "maximum-likelihood adjustment" because the term "least squares" corresponds only to the case of uncorrelated errors (i.e., Σ diagonal). His results may therefore be applied to the data-adjustment problem in this study.

The maximum-likelihood adjustment is to determine the most probable set of residuals $\bar{\gamma}$ that satisfy the condition equation (B4). Such a set is that which minimizes the quadratic form S appearing in the exponential term of (B6):

$$S=\bar{\gamma}^T\Sigma^{-1}\bar{\gamma} \tag{B7}$$

Solutions of constrained minima are especially suited to variational methods using Lagrangian multipliers. Multiplying the constraint equation (B4) by a row vector of constant multipliers $2\bar{\lambda}^T$ and subtracting this from (B7) give the expression to be minimized:

$$\bar{\gamma}^T\Sigma^{-1}\bar{\gamma}-2\bar{\lambda}^T(A\bar{\gamma}+\bar{\mathscr{E}})$$

Differentiating with respect to the only free variable $\bar{\gamma}$, and setting the result equal to zero,

$$(2\bar{\gamma}^T\Sigma^{-1}-2\bar{\lambda}^TA)d\bar{\gamma}=0$$

Since this equation must be satisfied for all variations $d\bar{\gamma}$,

$$\bar{\gamma}^T\Sigma^{-1}-\lambda^TA=0$$

Solving for $\bar{\gamma}$,

$$\bar{\gamma}=\Sigma A^T\bar{\lambda}$$

where

$$\Sigma^T=\Sigma\text{(symmetric)}$$

Substituting this expression into (B4), solving for $\bar{\lambda}$ and then substituting $\bar{\lambda}$ into the preceding equation give the desired expression for the residuals in terms of known quantities:

$$\bar{\gamma}=-\Sigma A^T(A\Sigma A^T)^{-1}\bar{\mathscr{E}} \tag{B8}$$

This result is then used in (B2) to obtain the most probable estimate of the adjusted quantities.

The second major result developed by Brown is the covariance matrix of the adjusted measurements. By comparing this matrix with Σ, one may gage the improvement due to the adjustment. Using equations (B3), (B4), (B5), and (B8), it can be shown that

$$\Sigma_{adj}=\Sigma-[(A\Sigma)^T(A\Sigma A^T)^{-1}(A\Sigma)] \tag{B9}$$

The bracketed term represents the covariance matrix of the residuals, and its diagonal elements should always be positive. Therefore, knowledge should always improve because of the adjustment.

It may be helpful to reiterate the preceding results at this time. The matrices and vectors required for data adjustment are A, Σ, and $\bar{\mathscr{E}}$. The matrix A is a predetermined constant for a given reference trajectory and time of fix. The discrepancy vector $\bar{\mathscr{E}}$ is evaluated at the measured conditions and thus depends in part upon random errors. The matrix Σ is in effect a measure of the accuracy of past and present knowledge before adjustment, and also of the correlation between derived measurements. What remains to be shown is how Σ is determined and how it is treated as the vehicle progresses towards the target planet.

Consider a general time t_n. The covariance matrix before adjustment $[\Sigma]_n$ is written in terms of its submatrices Σ_1, Σ_2, and Σ_3, where each of these submatrices corresponds to $\delta\bar{r}_A^{\circ}$, $\delta\bar{r}_{n-1}^{\circ}$, and $\delta\bar{r}_n^{\circ}$, respectively:

$$[\Sigma]_n=\begin{array}{|c|c|c|}\hline \Sigma_1 & \text{Correlation matrix} & 0 \\ \hline \text{Correlation matrix} & \Sigma_2 & 0 \\ \hline 0 & 0 & \Sigma_3 \\ \hline \end{array} \tag{B10}$$

The terms within the double square represent the accuracy of past knowledge, while Σ_3 represents the accuracy of current knowledge. Also, since $\delta\bar{r}_n^{\circ}$ is the current measurement and is independent of $\delta\bar{r}_A^{\circ}$ and $\delta\bar{r}_{n-1}^{\circ}$, the correlation matrices appearing in the third row and column are all zero.

The present derived measurement $\delta\bar{r}_n^{\circ}$ is a function of the primary measurements $(\delta\alpha_1, \delta\alpha_2, \delta\alpha_3, t)_n$ whose calibrated covariance matrix is diagonal and denoted Σ°:

$$\Sigma^{\circ}=\begin{bmatrix}\sigma_\alpha^2 & 0 & 0 & 0\\ 0 & \sigma_\alpha^2 & 0 & 0\\ 0 & 0 & \sigma_\alpha^2 & 0\\ 0 & 0 & 0 & \sigma_t^2\end{bmatrix} \tag{B11}$$

Now using equation (15), which relates the error $\Delta\bar{r}_n$ to the primary errors $(\Delta\alpha_i, t)_n$, the covariance matrix Σ_3 may be derived:

$$[\Sigma_3]_n = G_n \Sigma^\circ G_n^T \tag{B12}$$

where

$$G_n = \left[U^{-1} \;\vdots\; U^{-1}\bar{q} - \bar{v} \;\vdots \right]_n \tag{B13}$$

The submatrix Σ_2 and the correlation matrix are obtained directly from the adjusted covariance matrix of the previous navigation fix at t_{n-1}. For example,

$$[\Sigma_2]_n = [\Sigma_{3_{adj}}]_{n-1} \tag{B14}$$

The submatrix Σ_1 can also be obtained directly if no velocity correction was made at t_{n-1}:

$$[\Sigma_1]_n = [\Sigma_{1_{adj}}]_{n-1} \tag{B15}$$

If a correction was made at t_{n-1}, the effect of errors in applying Δv should be taken into account. If $\Delta\bar{r}$ is expressed in heliocentric spherical coordinates $(\Delta r, \theta, \phi)$ and it is assumed that the magnitude and directional errors are mutually independent, then the effect of $\Delta\bar{r}$ errors upon $\delta\bar{r}_A$ may be expressed as a covariance matrix Σ_1':

$$[\Sigma_1']_{n-1} = N \Sigma_{\Delta\bar{r}} N^T \tag{B16}$$

where

$$\left.\begin{aligned} \Sigma_{\Delta\bar{r}} &= \begin{bmatrix} \sigma^2_{\Delta r} & 0 & 0 \\ 0 & \sigma^2_\theta & 0 \\ 0 & 0 & \sigma^2_\varphi \end{bmatrix}_{n-1} \\ N &= B_{n-1} \begin{bmatrix} \dfrac{\Delta r_x}{\Delta r} & -\Delta r_y & \dfrac{\Delta r_x \Delta r_z}{\sqrt{\Delta r^2 - \Delta r_z^2}} \\ \dfrac{\Delta r_y}{\Delta r} & \Delta r_x & \dfrac{\Delta r_y \Delta r_z}{\sqrt{\Delta r^2 - \Delta r_z^2}} \\ \dfrac{\Delta r_z}{\Delta r} & 0 & \dfrac{-\Delta r^2}{\sqrt{\Delta r^2 - \Delta r_z^2}} \end{bmatrix} \end{aligned}\right\} \tag{B17}$$

and Σ_1 may now be written

$$[\Sigma_1]_n = [\Sigma_{1_{adj}}]_{n-1} + [\Sigma_1']_{n-1} \tag{B18}$$

The effect of a correctional maneuver is to reduce the accuracy of past knowledge; however, for reasonable errors in $\Delta\bar{r}$, results show that the effect is negligible because the diagonal elements of $\Sigma_1' \ll \Sigma_{1_{adj}}$. One may then conclude that the data-adjustment technique not only improves knowledge for a given navigation fix, but also improves knowledge with each successive fix.

The data-adjustment equations for the special case of $n=1$ are presented in appendix C.

APPENDIX C

SPECIAL CASE—FIRST CORRECTION

The guidance equations developed in appendix A and the text for the general condition t_n may be simplified for the first correction ($n=1$). In deriving the linearized guidance equations, it was assumed that the major cause of trajectory perturbations was velocity deviations at launch and not position deviations. The neglect of such position deviations was purely arbitrary but not necessary. With this assumption, $\delta\bar{r}_o=0$ and $\delta\bar{v}_0\equiv\delta v_L$.

BASIC GUIDANCE EQUATIONS

The velocity increment required to null the position deviation at arrival is found as follows: From equations (A10), (A11), and (A25),

$$\begin{aligned}\Delta\bar{v}_1&=Q_1^*\delta\bar{r}_1-\delta\bar{v}_1 \qquad \text{where } Q_1^*=V_1^*R_1^{*-1}\\ &=V_1^*R_1^{*-1}\delta\bar{r}_1-V_1\delta\bar{v}_L\\ &=(V_1^*R_1^{*-1}-V_1R_1^{-1})\delta\bar{r}_1\equiv H_1\delta\bar{r}_1 \qquad \text{(C1)}\end{aligned}$$

Since H_1 is predetermined, a measurement of the position deviation $\delta\bar{r}_1$ suffices to determine $\Delta\bar{v}_1$. Equation (C1) is equivalent to equation (A26) where $\delta\bar{r}_0=0$ and P_1 is undefined.

If the desired deviation at arrival is $\delta\bar{r}_A'$, then the velocity increment is obtained from equation (2):

$$\Delta\bar{v}_1=B_1^{-1}\delta\bar{r}_A'+H_1\delta\bar{r}_1 \qquad \text{(C2)}$$

The position deviation at arrival before the correction is applied is obtained from equation (1):

$$\delta\bar{r}_A=-B_1H_1\delta\bar{r}_1 \qquad \text{(C3)}$$

DATA-ADJUSTMENT EQUATIONS

When $n=1$, the residual equations (B2) become

$$\begin{aligned}\delta\bar{r}_A&=\delta\bar{r}_A^\circ+\bar{\gamma}_1\\ \delta\bar{r}_1&=\delta\bar{r}_1^\circ+\bar{\gamma}_2 \qquad \text{(C4)}\end{aligned}$$

Here, $\delta\bar{r}_A^\circ$ is the predicted value of the position deviation at arrival and is based on launch conditions as given by (A10):

$$\delta\bar{r}_A^\circ=R(t_A)\delta\bar{v}_L\equiv R_A\delta\bar{v}_L \qquad \text{(C5)}$$

Since, however, $\delta\bar{v}_L$ is normally distributed with a mean of zero, the best estimate for $\delta\bar{r}_A^\circ$ is also zero. Thus,

$$\delta\bar{r}_A^\circ=0 \qquad \text{(C6)}$$

The discrepancy vector given by (B3) becomes

$$\bar{\mathcal{E}}=B_1H_1\delta\bar{r}_1^\circ \qquad \text{(C7)}$$

and the (3×9) matrix, A, given by (B5) is reduced to a (3×6) matrix in partitioned form:

$$A=\left[\, I \;\vdots\; B_1H_1 \,\right] \qquad \text{(C8)}$$

The covariance matrix related to $\delta\bar{r}_A^\circ$ and $\delta\bar{r}_1^\circ$, which in general is given by equation (B10), reduces to a (6×6) matrix:

$$\Sigma_{n=1}=\begin{bmatrix}\Sigma_1 & \bigcirc\\ \bigcirc & \Sigma_3\end{bmatrix} \qquad \text{(C9)}$$

The submatrix Σ_3 is determined from (B12) and (B13) as usual. The submatrix Σ_1 is derived from (C5):

$$\Sigma_1=R_A\begin{bmatrix}\sigma_0^2 & 0 & 0\\ 0 & \sigma_0^2 & 0\\ 0 & 0 & \sigma_0^2\end{bmatrix}R_A^T \qquad \text{(C10)}$$

where σ_0 is the standard deviation of the injection-velocity error ($\delta\bar{v}_L$) components.

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TABLE I.—ASSUMED VALUES IN REFERENCE CASE

Parameter	Assumed value
I. Reference trajectory	
Launch date	Dec. 13, 1964
Trip time, days	192. 2
Relative velocity at Earth, ft/sec	12, 989
Eccentricity	0. 25404
Semimajor axis, A.U.	1. 3059
Inclination to ecliptic, deg	0. 556
Relative velocity at Mars, ft/sec	18, 385
Miss distance	0
II. Initial and measurement errors, σ, rms	
Injection velocity (x,y,z comp.), ft/sec	70
Angle error, sec arc	10
Clock error, %	0. 001
Δv magnitude error, %	0. 1
Δv direction error, sec arc	20
III. Guidance-logic factors	
Dead-band coefficient, k_{DB}	0. 5
Damping coefficient, k_{DM}	0. 5
Minimum Δv limit, Δv_{min}, ft/sec	10
Maximum Δv limit, Δv_{max}, ft/sec	1000

U.S. GOVERNMENT PRINTING OFFICE: 1962

NASA TR R-113
National Aeronautics and Space Administration.
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